Formalisation of Relational Algebra and a SQL-like Language with the RISCAL Model Checker

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January 26th, 2023

Outline

1. Topics
2. Recap on Relational Algebra
3. Implementation in RISCAL
4. Model Checking Algorithms and Theorems
5. A SQL-like wrapper language
6. Translation into standardized SQL
7. Current progress and further goals

Overview

- Formalization and verification of Relational Algebra with RISCAL
  - implementation of the common operations
  - model checking for proving correctness and theorems regarding the Relational Algebra
- Creation of a SQL-like language within RISCAL
- Demonstration by various examples

Recap on Relational Algebra

Definition

Let $n \in \mathbb{N}$ and $\mathcal{A} = \{A_1, \dots, A_n\}$ be an attribute schema. Then the set

$$\texttt{Row}_{\mathcal{A}} := \left\{ t : A \to \bigcup_{i=1}^{n} \mathrm{dom}(A_i) : t(A_i) \in \mathrm{dom}(A_i) \text{ for all } 1 \leq i \leq n \right\}$$

contains all tuples. Furthermore the set

$$\texttt{Relation}_{\mathcal{A}} := \{ r \subseteq \texttt{Row}_{\mathcal{A}} : |r| < \infty \}$$

contains all relations.

- Attributes are identifieres for columns. They are combined to get an attribute schema.
- Any attribute has a set of allowed values, a so called domain.
- The algebra consists of these types and some operations.

Recap on Relational Algebra

```
val M:ℕ; // maximum cardinality of relations
val N:ℕ; // maximum length of rows/tuples
val K:ℕ; // maximum number of tables
val D:ℕ; // maximum query depth
val L:ℕ; // maximum list length

type Element = ℕ[1];
type Attribute = ℕ[N-1];
type Length = ℕ[N];
type TableId = ℕ[K];
type Row = Map[Attribute, Element];
type Relation = Record[len:Length, tup:Set[Row]]
with |value.tup| ≤ M ∧ ∀ t:Row, i:Attribute. t ∈ value.tup ∧ i ≥ value.len ⇒ t[i] = 0;
type Database = Array[K, Relation];
```

Recap on Relational Algebra

- **Set operations**
- **Cartesian product**
- **Join:** Linking tables based on attributes
- **Selection:** Filtering on rows
- **Projection:** Filtering on columns

Implementation in RISCAL

Given an operator to implement:

1. Start off with a predicate representing the specification of an operation.
2. Rapid Prototyping: Choose a result that conforms to the predicate.
3. Algorithm: Implement it in terms of ...
    - RISCAL's high-level features like set constructors
    - and (only) loops, conditionals and variable assignments.

After the operations are verified we can check several theorems involving them.

Implementation in RISCAL

Definition

Let $\mathcal{A} = \{A_1, \dots, A_n\}$, $\mathcal{B} = \{B_1, \dots, B_m\}$ be relation schemas. The function $c : \texttt{Row}_{\mathcal{A}} \times \texttt{Row}_{\mathcal{B}} \to \texttt{Row}_{\mathcal{A} \cup \mathcal{B}}$ given by

$$c(t, u) := (t(A_1), \dots, t(A_n), u(B_1), \dots, u(B_m))$$

is called the concatenation of the rows t and u.

RISCAL

```
pred concat_spec(t:Row, t1:Row, t2:Row, n1:Length, n2:Length) ⇔
∀ i:Attribute. (
  if i < n1 then t[i] = t1[i]
  else if i ≥ n1 ∧ i < n1+n2 then t[i] = t2[i-n1]
  else t[i] = 0
);
```

Implementation in RISCAL

```
fun concat1(t1:Row, t2:Row, n1:Length, n2:Length):Row
requires n1 + n2 ≤ N;
= choose t:Row with concat_spec(t,t1,t2,n1,n2);
```

```
proc concat2(t1:Row, t2:Row, n1:Length, n2:Length):Row
requires n1 + n2 ≤ N;
ensures concat_spec(result,t1,t2,n1,n2); {
  var t:Row := Array[N,Element](0);
  for var i:Length:=0; i<n1; i:=i+1 do {
    t[i] := t1[i];
  }
  for var i:Length:=n1; i<n1+n2; i:=i+1 do {
    t[i] := t2[i-n1];
  }
  return t;
}
```

Implementation in RISCAL

Definition

Let $\mathcal{A} = \{A_1, \ldots, A_n\}$, $\mathcal{B} = \{B_1, \ldots, B_m\}$ be attribute schemas. The cartesian product $C : \texttt{Relation}_{\mathcal{A}} \times \texttt{Relation}_{\mathcal{B}} \to \texttt{Relation}_{\mathcal{A} \cup \mathcal{B}}$ is given by

$$C(r, s) := \{c(t, u) : t \in r \wedge u \in s\}.$$

RISCAL

```
pred cartesian_spec(r:Relation, r1:Relation, r2:Relation) ⇔
r.len = r1.len+r2.len ∧
∀ t:Row. t∈r.tup ⇔ ∃ t1:Row, t2:Row.
  t1∈r1.tup ∧ t2∈r2.tup ∧ t = concat1(t1,t2,r1.len,r2.len);
```

Implementation in RISCAL

```
fun cartesian1(r1:Relation, r2:Relation):Relation
requires r1.len+r2.len ≤ N ∧ |r1.tup|*|r2.tup| ≤ M;
  = choose r:Relation with cartesian_spec(r,r1,r2);
```

```
fun cartesian2(r1:Relation, r2:Relation):Relation
requires r1.len+r2.len ≤ N ∧ |r1.tup|*|r2.tup| ≤ M;
ensures cartesian_spec(result,r1,r2);
  = ⟨len: r1.len+r2.len, tup: {concat1(t1,t2,r1.len,r2.len) | t1∈r1.tup, t2∈r2.tup}⟩;
```

```
proc cartesian3(r1:Relation, r2:Relation):Relation
requires r1.len+r2.len ≤ N ∧ |r1.tup|*|r2.tup| ≤ M;
ensures cartesian_spec(result,r1,r2); {
  var q:Relation := ⟨len: r1.len+r2.len, tup: choose s:Set[Row] with |s|=0⟩;

  for t1 ∈ r1.tup do {
    for t2 ∈ r2.tup do {
      q.tup := q.tup ∪ {concat2(t1, t2, r1.len, r2.len)};
    }
  }

  return q;
}
```

Implementation in RISCAL

Definition

Let $\mathcal{A} = \{A_1, \ldots, A_n\}$, $\mathcal{B} = \{B_1, \ldots, B_m\}$ be attribute schemas. The function $\bowtie$: $\texttt{Relation}_{\mathcal{A}} \times \texttt{Relation}_{\mathcal{B}} \times \mathcal{A} \times \mathcal{B} \to \texttt{Relation}_{\mathcal{A} \cup \mathcal{B}}$ given by

$$\bowtie (r, s, A, B) := \{c(t, u) : t \in r \wedge u \in s \wedge t(A) = u(B)\}$$

is called the (equi-)join of r and s on A and B.

RISCAL

```
pred join_spec(s:Relation, r1:Relation, r2:Relation, n1:Attribute, n2:Attribute) ⇔
s.len = r1.len+r2.len ∧
∀ t:Row. t∈s.tup ⇔ ∃ t1:Row, t2:Row.
  (t1∈r1.tup ∧ t2∈r2.tup ∧ t = concat1(t1,t2,r1.len,r2.len) ∧ t1[n1] = t2[n2]);
```

Implementation in RISCAL

```
fun join1(r1:Relation, r2:Relation, n1:Attribute, n2:Attribute):Relation
requires n1<r1.len ∧ n2<r2.len ∧ r1.len+r2.len ≤ N ∧ |r1.tup|*|r2.tup| ≤ M;
  = choose s:Relation with join_spec(s,r1,r2,n1,n2);
```

```
fun join2(r1:Relation, r2:Relation, n1:Attribute, n2:Attribute):Relation
requires n1<r1.len ∧ n2<r2.len ∧ r1.len+r2.len ≤ N ∧ |r1.tup|*|r2.tup| ≤ M;
ensures join_spec(result,r1,r2,n1,n2);
  = ⟨len: r1.len+r2.len, tup:
    {concat1(t1,t2,r1.len,r2.len) | t1∈r1.tup, t2∈r2.tup with t1[n1] = t2[n2]}⟩;
```

```
proc join3(r1:Relation, r2:Relation, n1:Attribute, n2:Attribute):Relation
requires n1<r1.len ∧ n2<r2.len ∧ r1.len+r2.len ≤ N ∧ |r1.tup|*|r2.tup| ≤ M;
ensures join_spec(result,r1,r2,n1,n2); {
  var q:Relation := ⟨len: r1.len+r2.len, tup: choose s:Set[Row] with |s|=0⟩;

  for t1 ∈ r1.tup do {
    for t2 ∈ r2.tup do {
      if t1[n1] = t2[n2] then {
        q.tup := q.tup ∪ {concat2(t1, t2, r1.len, r2.len)};
      }
    }
  }

  return q;
}
```

Implementation in RISCAL

Definition

Let $\mathcal{A} = \{A_1, \ldots, A_n\}$ be an attribute schema. Then the function $\sigma : \texttt{Relation}_{\mathcal{A}} \times \mathcal{A} \times \bigcup_{A \in \mathcal{A}} \mathrm{dom}(A) \rightarrow \texttt{Relation}_{\mathcal{A}}$ given by

$$\sigma(r, A, a) := \{t \in r : t(A) = a\}$$

is called the selection of all rows t in r with value a at attribute A.

RISCAL

```
pred select_spec(s:Relation, r:Relation, a:Attribute, e:Element) ⇔
s.len = r.len ∧ ∀ t:Row. t∈s.tup ⇔ t∈r.tup ∧ t[a] = e;
```

Implementation in RISCAL

```
fun select1(r:Relation, a:Attribute, e:Element):Relation
requires a < r.len;
  = choose s:Relation with select_spec(s,r,a,e);
```

```
fun select2(r:Relation, a:Attribute, e:Element):Relation
requires a < r.len;
ensures select_spec(result,r,a,e);
  = ⟨len: r.len, tup: {t | t∈r.tup with t[a] = e}⟩;
```

```
proc select3(r:Relation, a:Attribute, e:Element):Relation
requires a < r.len;
ensures select_spec(result,r,a,e); {
  var q:Relation := ⟨len: r.len, tup: choose s:Set[Row] with |s|=0⟩;

  for t ∈ r.tup do {
    if t[a] = e then {
      q.tup = q.tup ∪ {t};
    }
  }
  return q;
}
```

Implementation in RISCAL

Definition

Let $\mathcal{A} = \{A_1, \ldots, A_n\}$ be an attribute schema. The operation $\pi : \texttt{Relation}_{\mathcal{A}} \times \mathcal{P}(\mathcal{A}) \to \bigcup_{\mathcal{A}' \subseteq \mathcal{A}} \texttt{Relation}_{\mathcal{A}'}$ defined by

$$\pi(r, \mathcal{B}) = \{t|_{\mathcal{B}} : t \in r\}$$

is called projection of r on $\mathcal{B}$.

RISCAL

```
pred project_spec(s:Relation, r:Relation, columns:Array[N,Length]) ⇔
s.len = |{i | i:Attribute with columns[i] ≠ N}| ∧ (∀ tr:Row. tr∈r.tup ⇒
  ∃ ts:Row. ts∈s.tup ∧ ∀ i:Attribute. i < s.len ⇒ ts[i]=tr[columns[i]]);
```

Implementation in RISCAL

```
fun project1(r:Relation, columns:Array[N,Length]):Relation
requires (∃ i:Attribute. ∀ j:Attribute. (j>i ⇒ columns[j] = N) ∧ (j≤i ⇒ columns[j] < r.
      len));
  = choose s:Relation with project_spec(s,r,columns);
```

```
proc project2(r:Relation, columns:Array[N,Length]):Relation
requires (∃ i:Attribute. ∀ j:Attribute. (j>i ⇒ columns[j] = N) ∧ (j≤i ⇒ columns[j] < r.
      len));
ensures project_spec(result,r,columns); {

  var l:Length := |{i | i:Attribute with columns[i] ≠ N}|;
  var q:Relation := ⟨len: l, tup: choose s:Set[Row] with |s|=0⟩;

  for t ∈ r.tup do {
    var tn:Row := Array[N,Element](0);

    var j:Length := 0;
    for var i:Length := 0; i<N; i:=i+1 do {
      if columns[i] ≠ N then {
        tn[j] := t[columns[i]];
        j := j+1;
      }
    }
    q.tup := q.tup ∪ {tn};
  }

  return q;
}
```

Model Checking Algorithms and Theorems

- For each function/procedure:
  - execute the function and
  - check the postcondition for every valid input
- For each theorem: Check validity

Both tasks can be accomplished by the RISCAL task "Execute operation".

Theorem 1

Let $\mathcal{A}$ be an attribute schema, $A \in \mathcal{A}$, $a \in \mathrm{dom}(A)$ and $r, s \in \texttt{Relation}_{\mathcal{A}}$ relations. Then

$$\sigma(r \gamma s, A, a) = \sigma(r, A, a) \gamma \sigma(s, A, a)$$

holds for all $\gamma \in \{\cup, \cap, \setminus\}$.

```
theorem select_union_equiv(r1:Relation, r2:Relation, a:Attribute, e:Element)
requires a < r1.len ∧ a < r2.len ∧ union_compatible(r1,r2) ∧ |r1.tup| + |r2.tup| ≤ M; ⇔
select2(rUnion(r1,r2),a,e) = rUnion(select2(r1, a, e), select2(r2, a, e));

theorem select_intersect_equiv(r1:Relation, r2:Relation, a:Attribute, e:Element)
requires a < r1.len ∧ a < r2.len ∧ union_compatible(r1,r2); ⇔
select2(rIntersect(r1,r2),a,e) = rIntersect(select2(r1, a, e), select2(r2, a, e));

theorem select_minus_equiv(r1:Relation, r2:Relation, a:Attribute, e:Element)
requires a < r1.len ∧ a < r2.len ∧ union_compatible(r1,r2); ⇔
select2(rMinus(r1,r2),a,e) = rMinus(select2(r1, a, e), select2(r2, a, e));
```

Theorem 2

Let $\mathcal{A}$ be an attribute schema, $A, B \in \mathcal{A}$, $a \in \mathrm{dom}(A), b \in \mathrm{dom}(B)$ and $r \in \texttt{Relation}_{\mathcal{A}}$ a relation. Then

$$\sigma(\sigma(r, B, b), A, a) = \sigma(r, A, a) \cap \sigma(r, B, b)$$

holds.

```
theorem select_intersect_comp(r:Relation, a:Attribute, e:Element, b:Attribute, f:Element)
requires a < r.len ∧ b < r.len; ⇔
select2(select2(r, a, e), b, f) = rIntersect(select2(r, a, e), select2(r, b, f));
```

Model Checking Algorithms and Theorems

Theorem 3

Let $\mathcal{A} = \{A_1, \ldots, A_n\}$ be an attribute schema, $A, B \in \mathcal{A}$ and $r \in \texttt{Relation}_{\mathcal{A}}$ a relation. Then

$$\pi(r, \{A, B\}) \subseteq C(\pi(r, \{A\}), \pi(r, \{B\}))$$

holds.

```
theorem project_cartesian_subset(r:Relation, a:Attribute, b:Attribute)
requires a < r.len ∧ b < r.len ∧ 2*|r.tup| ≤ M; ⇔
project2(r,attributes(List!node(a, List!node(b, List!nil)))).tup
  ⊆ cartesian2(project2(r, attributes(List!node(a, List!nil))),
          project2(r, attributes(List!node(b, List!nil)))).tup;
```

Theorem 3

Let $\mathcal{A}, \mathcal{B}$ be attribute schemas, $A \in \mathcal{A}$, $B \in \mathcal{B}$ and $r \in \texttt{Relation}_{\mathcal{A}}$, $s \in \texttt{Relation}_{\mathcal{A}}$ relations. Then

$$\bowtie (r, s, A, B) \subseteq C(r, s)$$

holds.

```
theorem join_cartesian_subset(r1:Relation, r2:Relation, n1:Attribute, n2:Attribute)
requires n1<r1.len ∧ n2<r2.len ∧ r1.len+r2.len ≤ N ∧ |r1.tup|*|r2.tup| ≤ M; ⇔
join3(r1, r2, n1, n2).tup ⊆ cartesian3(r1, r2).tup;
```

An SQL-like wrapper language

Grammar

$$
\begin{array}{lcl}
L & \in & \mathit{List} \\
Q & \in & \mathit{Query} \\
L & ::= & () \mid (A, L) \\
Q & ::= & \texttt{from}(T) \\
 & \mid & \texttt{on}(A_1, A_2, Q_1, Q_2) \\
 & \mid & \texttt{where}(A, E, Q) \\
 & \mid & \texttt{select}(L, Q) \\
 & \mid & \texttt{add}(Q_1, Q_2) \\
 & \mid & \texttt{inters}(Q_1, Q_2) \\
 & \mid & \texttt{minus}(Q_1, Q_2) \\
 & \mid & \texttt{cart}(Q_1, Q_2)
\end{array}
$$

- The natural variables T and A identify tables and attributes, whereas E denotes either 0 or 1.
- The language $\texttt{SQL}_{\text{min}}$ consists of all expressions in the syntactical domain `Query`.

An SQL-like wrapper language

```
rectype(L) List = nil | node(Attribute, List);
rectype(D) Query = from(TableId)
          | on(Attribute, Attribute, Query, Query)
          | where(Attribute, Element, Query)
          | select(List, Query)
          | add(Query, Query)
          | inters(Query, Query)
          | minus(Query, Query)
          | cart(Query, Query);
```

An SQL-like wrapper language

Definition

Let $\mathrm{db} : \mathtt{TableId} \to \bigcup_{i=1}^{n} \mathtt{Relation}_{\mathcal{A}_i}$ be a database. We define the denotational semantics $[\![\cdot]\!]_{\mathrm{db}} : \mathit{Query} \to \mathtt{Relation}_{\mathcal{A}}$ of $\mathtt{SQL}_{\mathsf{min}}$ by recursive induction in the following way:

$$
\begin{aligned}
[\![\mathtt{from}(T)]\!]_{\mathrm{db}} &:= \mathrm{db}(T) \\
[\![\mathtt{on}(A_1, A_2, Q_1, Q_2)]\!]_{\mathrm{db}} &:= \bowtie([\![Q_1]\!]_{\mathrm{db}}, [\![Q_2]\!]_{\mathrm{db}}, A_1, A_2) \\
[\![\mathtt{where}(A, E, Q)]\!]_{\mathrm{db}} &:= \sigma([\![Q]\!]_{\mathrm{db}}, A, E) \\
[\![\mathtt{select}(L, Q)]\!]_{\mathrm{db}} &:= \pi([\![Q]\!]_{\mathrm{db}}, L) \\
[\![\mathtt{add}(Q_1, Q_2)]\!]_{\mathrm{db}} &:= [\![Q_1]\!]_{\mathrm{db}} \cup [\![Q_2]\!]_{\mathrm{db}} \\
[\![\mathtt{inters}(Q_1, Q_2)]\!]_{\mathrm{db}} &:= [\![Q_1]\!]_{\mathrm{db}} \cap [\![Q_2]\!]_{\mathrm{db}} \\
[\![\mathtt{minus}(Q_1, Q_2)]\!]_{\mathrm{db}} &:= [\![Q_1]\!]_{\mathrm{db}} \backslash [\![Q_2]\!]_{\mathrm{db}} \\
[\![\mathtt{cart}(Q_1, Q_2)]\!]_{\mathrm{db}} &:= C([\![Q_1]\!]_{\mathrm{db}}, [\![Q_2]\!]_{\mathrm{db}})
\end{aligned}
$$

An SQL-like wrapper language

```
proc query(db:Database, q:Query):Relation {
  var r:Relation := match q with {
    from(tid:TableId) -> db[tid];
    on(n1:Attribute, n2:Attribute, q1:Query, q2:Query) -> join(query(db, q1), query(db,
         q2), n1, n2);
    where(a:Attribute, e:Element, q:Query) -> select(query(db, q), a, e);
    select(a:List, q:Query) -> project(query(db, q), attributes(a));
    add(q1:Query, q2:Query) -> rUnion(query(db, q1), query(db, q2));
    inters(q1:Query, q2:Query) -> rIntersect(query(db, q1), query(db, q2));
    minus(q1:Query, q2:Query) -> rMinus(query(db, q1), query(db, q2));
    cart(q1:Query, q2:Query) -> cartesian(query(db, q1), query(db, q2));
  };
  return r;
}
```

Translation into standardized SQL

A SQL query is always executed in the same order, i.e. we can sort the clauses in the way

$$\underset{4}{\texttt{FROM}} \succ \underset{3}{\texttt{ON}} \succ \underset{2}{\texttt{WHERE}} \succ \underset{1}{\texttt{SELECT}},$$

where $C_1 \succ C_2$ means that clause C_1 is evaluated before clause C_2. If the corresponding operations of $\texttt{SQL}_{\mathsf{min}}$ are composed in this order, they are equivalent to a single SQL query without subqueries.

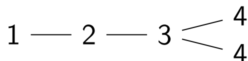
Translation into standardized SQL

We can list every possibility, because

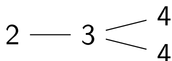
- the leaves of any AST must contain `FROM` clauses and
- under the assumption that the tree nodes of an AST contain clauses of strictly ascending priority, there are maximal 4 layers.

This equates to the eigth "atomic" SQL expressions.

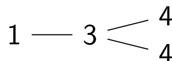
Translation into standardized SQL




```
SELECT <attributes>
FROM <table>
JOIN <table>
ON <attribute> = <attribute>
WHERE <attribute> = <value>;
```

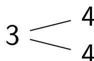



```
SELECT *
FROM <table>
JOIN <table>
ON <attribute> = <attribute>
WHERE <attribute> = <value>;
```

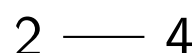



```
SELECT <attributes>
FROM <table>
JOIN <table>
ON <attribute>
  = <attribute>;
```

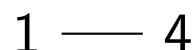
```
SELECT <attributes>
FROM <table>
WHERE <attribute> = <value>;
```




```
SELECT *
FROM <table>
JOIN <table>
ON <attribute> = <attribute>;
```




```
SELECT *
FROM <table>
WHERE <attribute>
= <value>;
```




```
SELECT <attributes>
FROM <table>;
```

```
SELECT *
FROM <table>;
```

Translation into standardized SQL

```
proc query_to_sql(q0:Query):()
proc translate_on(n1:Attribute, n2:Attribute, q0_1:Query, q0_2:Query):()
proc translate_cart(q0_1:Query, q0_2:Query):()
proc translate_where(a:Attribute, e:Element, q0:Query):()
proc translate_from(tid:TableId):()
proc translate_blank(q0:Query):()
proc translate_select(l:List, q0:Query):()
```

Translation into standardized SQL

```
proc query_to_sql(q0:Query):() {
  match q0 with {
    add(q1_1:Query, q1_2:Query) -> {
      translate_add(q1_1, q1_2);
    }
    inters(q1_1:Query, q1_2:Query) -> {
      translate_inters(q1_1, q1_2);
    }
    minus(q1_1:Query, q1_2:Query) -> {
      print "Not implemented";
    }
    cart(q1_1:Query, q1_2:Query) -> {
      translate_cart(q1_1, q1_2);
    }
    select(l:List, q1:Query) -> {
      translate_select(l, q1);
    }
    where(a:Attribute, e:Element, q1:Query) -> {
      print "SELECT *";
      translate_where(a, e, q1);
    }
    on(n1:Attribute, n2:Attribute, q1_1:Query, q1_2:Query) -> {
      print "SELECT *";
      translate_on(n1, n2, q1_1, q1_2);
    }
    from(tid:TableId) -> {
      print "SELECT *";
      translate_from(tid);
    }
  }
}
```

Translation into standardized SQL

<table>
<tr><th>RISCAL</th><th>query_to_sql</th><th>Sqlite</th></tr>
<tr><td>

```
Query!on(0,0,
  Query!select(
    List!node(0,
      List!node(2, List!nil)),
    Query!where(1,1,
      Query!from(0)
    )
  ),
  Query!from(1)
);
```

</td><td>

```
SELECT * FROM (
  SELECT 0, 2
  FROM 0
  WHERE 1 = 1
)
JOIN 1
ON 0 = 0
```

</td><td>

```
SELECT * FROM (
  SELECT "0"."0", "0"."2"
  FROM "0"
  WHERE "0"."1" = 1
) AS "_0"
JOIN "1"
ON "_0"."0" = "1"."0";
```

</td></tr>
</table>

Current progress and further goals

Done:

- Formalization and verification with a Model Checker
- Design of a Language

Work in progress:

- Theorem proving with RISCTP
- Various formal improvements