SMT SOLVING: DECIDABLE THEORIES

Course “Computational Logic”

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Theories

- A theory T is a set of first-order sentences (closed formulas) that is closed under logical consequence:

$$T \models F \text{ if and only if } F \in T, \text{ for every first-order formula } F.$$

- T may be defined as the set $Th(\mathcal{M}) := \{F \mid \forall M \in \mathcal{M}.\ M \models F\}$ of all sentences that hold in (every element of) some class $\mathcal{M}$ of structures.
  - Notation $Th(\mathbb{N}, 0, 1, +, \cdot, \leq)$: the theory where $0, 1, +, \cdot, \leq$ are interpreted as the usual natural number constants, functions, predicates.
- T may be also defined as the set $Cn(A) := \{F \mid A \models F\}$ of consequences of some recursively enumerable set A of first-order formulas called axioms.
  - A set is recursively enumerable if a machine can produce a list of its elements.
  - If $T = Cn(A)$ for some (finite) set A, then T is (finitely) axiomatiziable.
  - Gödel's incompleteness theorem: $Th(\mathbb{N}, 0, 1, +, \cdot, \leq)$ is <u>not</u> axiomatizable.

A theory describes a "domain of interest".

Decision Problems

Theories give rise to two related decision problems.

- The problem of Validity Modulo Theories:
  - Given: a first-order formula F and a first-order theory T.
  - Decide: does $T \models F$ hold, i.e., is F is a logical consequence of T?
- The problem of Satisfiability Modulo Theories (SMT):
  - Given: a first-order formula F and a first-order theory T.
  - Decide: is $T \cup \{F\}$ satisfiable?
- Duality: $T \models F$ if and only if $T \cup \{\neg F\}$ is not satisfiable.

An SMT solver is a decision procedure for the SMT problem (with respect to some theory or combination of theories); thus it also decides the dual validity problem.

Decidable Problems

For certain classes of formulas/theories, the satisfiability problem is decidable.

- Prenex normal form $\forall^n \exists^m$ (validity) or $\exists^n \forall^m$ (satisfiability) (“AE/EA fragment”).
- Formulas without functions and with only unary predicates (“monadic fragment”).
- Every theory over finite domains (e.g., the domain of fixed-size bit vectors).
- Quantifier-free theory of equality with uninterpreted functions (“equational logic”).
- Theory of arrays, theory of recursive data structures.
- Linear arithmetic over integers (“Presburger arithmetic”), natural numbers, reals.
- Theory of reals (“elementary algebra”), complex numbers, algebraically closed fields.
- Logical consequences of equalities over groups, rings, fields (“word problems”).
- ...

As we will see later, also any combination of decidable theories is decidable.

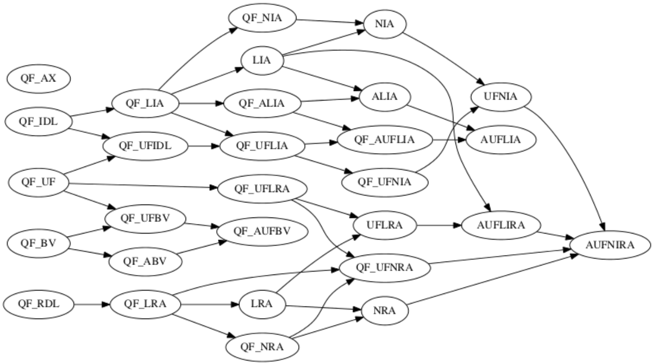
SMT-LIB: The Satisfiability Modulo Theories Library

http://smt-lib.org

- A library of theories/logics of practical relevance.
- A common input language for SMT solvers.
- A repository of benchmarks.
- The basis of the yearly SMT-COMP competition.
    - https://smt-comp.github.io

Many automated/interactive reasoners and program verifiers are equipped with SMT-LIB interfaces to external SMT solvers.

The SMT-LIB Library



- QF_UF: Unquantified formulas built over a signature of uninterpreted (i.e., free) sort and function symbols.
- QF_LIA: Unquantified linear integer arithmetic. In essence, Boolean combinations of inequations between linear polynomials over integer variables.

Not every logic is decidable, e.g., NIA (non-linear integer arithmetic).

Z3: An SMT solver with SMT-LIB Support

Software: `https://github.com/Z3Prover/z3`

Tutorial: Z3 — A Tutorial (Leonardo de Moura and Nikolaj Bjørner)

- An SMT solver developed since 2007 at Microsoft Research.
  - Nikolaj Bjørner and Leonardo de Moura.
  - Open source since 2015 under the MIT License.
- Highly efficient and versatile.
  - Frequent winner of various divisions of the SMT-COMP series.
  - Backend of various software verification systems (e.g., Microsoft Boogie).
- Uses the SMT-LIB language and supports various SMT-LIB logics.
  - Uninterpreted functions, linear arithmetic, fixed-size bit-vectors, algebraic datatypes, arrays, polynomial arithmetic, ...
- Also supports quantification.
  - However, when using quantifiers, the solver is generally incomplete.

Z3 gradually evolves into a full-fledged automated theorem prover.

The SMT-LIB Language

```
; file example1.smt2: Integer arithmetic
(set-logic QF_LIA)
(declare-const x Int)
(declare-const y Int)
(assert (= (- x y) (+ x (- y) 1)))
(check-sat)
(exit)

debian10!1> z3 example1.smt
unsat
```

```
; file example2.smt2: Getting values or models
(set-logic QF_LIA)
(declare-const x Int)
(declare-const y Int)
(assert (= (+ x (* 2 y)) 20))
(assert (= (- x y) 2))
(check-sat)
(get-value (x y))
(get-model)
(exit)

debian10!1> z3 example2.smt2
sat
((x 8) (y 6))
(model
  (define-fun y () Int 6)
  (define-fun x () Int 8)
)
```

The SMT-LIB Language

```
; file example3.smt2:
; Modeling sequential code in SSA form
; Buggy swap: int x, y; int t = x; x = y; y = x;
(set-logic QF_UFLIA)

(declare-fun x (Int) Int)
(declare-fun y (Int) Int)
(declare-fun t (Int) Int)
(assert (= (t 0) (x 0)))
(assert (= (x 1) (y 0)))
(assert (= (y 1) (x 1)))
(assert (not
  (and (= (x 1) (y 0))
       (= (y 1) (x 0)))))

(check-sat)
(get-value ((x 0) (y 0) (x 1) (y 1)))
(get-model)
(exit)
```

```
sat
(((x 0) 2)
 ((y 0) 3)
 ((x 1) 3)
 ((y 1) 3))
(model
  (define-fun y ((x!1 Int)) Int
    (ite (= x!1 0) 3
    (ite (= x!1 1) 3
      3)))
  (define-fun t ((x!1 Int)) Int
    (ite (= x!1 0) 2
      2))
  (define-fun x ((x!1 Int)) Int
    (ite (= x!1 0) 2
    (ite (= x!1 1) 3
      2)))
)
```

Example Application: Program Verification

We can reduce the verification of programs to deciding the satisfiability of formulas.

- Verification of program with respect to pre- and post-condition:

  $\{a[0] = x \wedge a[1] = y \wedge a[2] = z\}$

  ```
  i = 0;   m = a[i];
  i = i+1; if (a[i] < m) m = a[i];
  i = i+1; if (a[i] < m) m = a[i];
  ```

  $\{m \leq x \wedge m \leq y \wedge m \leq z \wedge (m = x \vee m = y \vee m = z)\}$

- Satisfiability of formula:

$$\begin{aligned}
& a[0] = x \wedge a[1] = y \wedge a[2] = z \wedge \\
& i_0 = 0 \wedge m_0 = a[i_0] \wedge \\
& i_1 = i_0 + 1 \wedge (\text{if } a[i_1] < m_0 \text{ then } m_1 = a[i_1] \text{ else } m_1 = m_0) \wedge \\
& i_2 = i_1 + 1 \wedge (\text{if } a[i_2] < m_1 \text{ then } m_2 = a[i_2] \text{ else } m_2 = m_1) \wedge \\
& \neg(m_2 \leq x \wedge m_2 \leq y \wedge m_2 \leq z \wedge (m_2 = x \vee m_2 = y \vee m_2 = z))
\end{aligned}$$

The unsatisfiability of the formula establishes the correctness of the program with respect to its specification; a satisfying valuation determines a violating program run.

Program Verification: SMT-LIB Script

```
; file minimum.smt2:
(set-logic QF_UFLIA)

(declare-fun a (Int) Int)
(declare-const x Int) (declare-const y Int) (declare-const z Int)
(declare-const i0 Int) (declare-const i1 Int) (declare-const i2 Int)
(declare-const m0 Int) (declare-const m1 Int) (declare-const m2 Int)

(assert (= (a 0) x)) (assert (= (a 1) y)) (assert (= (a 2) z))
(assert (= i0 0)) (assert (= m0 (a i0)))
(assert (= i1 (+ i0 1))) (assert (ite (< (a i1) m0) (= m1 (a i1)) (= m1 m0)))
(assert (= i2 (+ i1 1))) (assert (ite (< (a i2) m1) (= m2 (a i2)) (= m2 m1)))
(assert (not
  (and (and (and (<= m2 x) (<= m2 y)) (<= m2 z))
       (or  (or  (=  m2 x) (=  m2 y)) (=  m2 z)))))

(check-sat) (exit)

debian10!1> z3 minimum.smt2
unsat
```

Program Verification: SMT-LIB Script

```
; file minimum2.smt2:
...
; BUG: ">" rather than "<"
(assert (ite (> (a i2) m1) (= m2 (a i2)) (= m2 m1)))
...
(check-sat) (get-value (x y z i0 m0 i1 m1 i2 m2)) (get-model) (exit)

alan!89> z3 minimum2.smt2
sat
((x 1) (y 0) (z 2) (i0 0) (m0 1) (i1 1) (m1 0) (i2 2) (m2 2))
(model
  (define-fun m0 () Int 1) (define-fun i1 () Int 1) (define-fun m2 () Int 2)
  (define-fun y () Int 0) (define-fun m1 () Int 0) (define-fun i2 () Int 2)
  (define-fun i0 () Int 0) (define-fun z () Int 2) (define-fun x () Int 1)
  (define-fun a ((x!1 Int)) Int (ite (= x!1 0) 1 (ite (= x!1 1) 0 (ite (= x!1 2) 2 1)))))
```

The assignments of a buggy program with an inverted test operation.

The Theory LRA: Linear Real Arithmetic

Essentially the SMT-LIB logic QF_LRA.

- LRA is a quantifier-free first-order theory.
  - Interpretation over the domain $\mathbb{R}$ of real numbers.
  - Only atomic formulas are inequalties $a \leq b$ with polynomials a, b.
    - Integer and rational constants, functions + and $\cdot$, predicate $\leq$.
    - Also $-, <, >, \geq, =$ are allowed: $a - b$ can be reduced to $a + (-1) \cdot b$; $\{<, >\}$ can be reduced to $\{=, \leq, \geq\}$; $=$ can be reduced to $\{\leq, \geq\}$; $\geq$ can be reduced to $\leq$.
  - Linear: in every multiplication $a \cdot b$, a must be a constant.
- LRA-Satisfiability of formula F:
  - Convert F into its disjunctive normal form $C_1 \vee \ldots \vee C_n$.
  - F is LRA-satisfiable if and only if some C_i is LRA-satisfiable.

To decide the LRA-Satisfiability of F, it suffices to decide the satisfiability of a conjunction of (possibly negated) inequalities $a \leq b$ with linear polynomials a, b (in the following, we only consider conjunctions of unnegated inequalities).

Deciding LRA-Satisfiability by Fourier-Motzkin Elimination

Joseph Fourier (1826), Theodore Motzkin (1936).

function FOURIERMOTZKIN(F) ▷ F is a conjunction of inequalities $a \leq b$ with linear polynomials a, b
  while F contains a variable **do**
    Choose some variable x in F
    Arithmetically transform every inequality in which x occurs into the form $a \leq x$ or $x \leq b$
    Let A be the set of all a where $a \leq x$ is an inequality in F.
    Let B be the set of all b where $x \leq b$ is an inequality in F.
    Remove from F all inequalities of form $a \leq x$ and $x \leq b$.
    Add to F a (possibly simplified version of the) inequality $a \leq b$ for every pair $(a, b) \in A \times B$
  end while
  if F contains a constraint $c_1 \leq c_2$ with constant c_1 greater than constant c_2 **then**
    return false ▷ unsatisfiabile
  else
    return true ▷ satisfiable
  end if
end function

Example

LRA-Satisfiability of formula $F :\Leftrightarrow (z \leq x - y) \wedge (x + 2 \cdot y \leq 5) \wedge (y \leq 4 \cdot z - 2 \cdot x)$

- Eliminate x:
  - Transform: $(z + y \leq x) \wedge (x \leq 5 - 2 \cdot y) \wedge (x \leq 2 \cdot z - \frac{1}{2} \cdot y)$
  - Eliminate: $(z + y \leq 5 - 2 \cdot y) \wedge (z + y \leq 2 \cdot z - \frac{1}{2} \cdot y)$
  - Simplify: $(z \leq 5 - 3 \cdot y) \wedge (\frac{3}{2} \cdot y \leq z)$
- Eliminate z:
  - Transform: $(\frac{3}{2} \cdot y \leq z) \wedge (z \leq 5 - 3 \cdot y)$
  - Eliminate: $(\frac{3}{2} \cdot y \leq 5 - 3 \cdot y)$
  - Simplify: $(\frac{9}{2} \cdot y \leq 5)$
- Eliminate y:
  - Transform: $(y \leq \frac{10}{9})$
  - Eliminate: $\top$

F is LRA-satisfiable (by, e.g., $y := 0 \in [-\infty, \frac{10}{9}], z := 0 \in [0, 5], x := 0 \in [0, 0]$).

Example

LRA-Satisfiability of formula $F :\Leftrightarrow (x \leq y) \wedge (x \leq z) \wedge (y + 2 \cdot z \leq x) \wedge (1 \leq x)$

- Eliminate x:
  - Transform: $(y + 2 \cdot z \leq x) \wedge (1 \leq x) \wedge (x \leq y) \wedge (x \leq z)$
  - Eliminate: $(y + 2 \cdot z \leq y) \wedge (y + 2 \cdot z \leq z) \wedge (1 \leq y) \wedge (1 \leq z)$
  - Simplify: $(z \leq 0) \wedge (y + z \leq 0) \wedge (1 \leq y) \wedge (1 \leq z)$
- Eliminate z:
  - Transform: $(1 \leq z) \wedge (z \leq 0) \wedge (z \leq -y) \wedge (1 \leq y)$
  - Eliminate: $(1 \leq 0) \wedge (1 \leq -y) \wedge (1 \leq y)$
  - Simplify: $(1 \leq 0) \wedge (y \leq -1) \wedge (1 \leq y)$
- Eliminate y:
  - Transform: $(1 \leq y) \wedge (y \leq -1) \wedge (1 \leq 0)$
  - Eliminate: $(1 \leq -1) \wedge (1 \leq 0)$

F is LRA-unsatisfiable.

The Theory EUF: Equality with Uninterpreted Functions

Essentially the SMT-LIB logic QF_UF.

- EUF is a quantifier-free first-order theory with only predicate "$=$".
  - Syntax: an arbitrary propositional combination of equalities.
  - Semantics: the fixed interpretation of "$=$" as "equality".
- EUF is sufficient to also deal with arbitrary other predicates in a formula F:
  - Introduce a fresh constant T and a fresh function f_p for every other predicate p.
  - Transform every atomic formula $p(\ldots)$ into an equality $f_p(\ldots) = T$.
  - Formula F is satisfiable if and only if its transformed version is EUF-satisfiable.
- EUF-satisfiability of formula F:
  - Convert F into its disjunctive normal form $C_1 \vee \ldots \vee C_n$.
  - F is EUF-satisfiable if and only if some C_i is EUF-satisfiable.

It suffices to decide the satisfiability of a conjunction of (negated) equalities.

Deciding EUF-Satisfiability by Congruence Closure

Greg Nelson and Derek C. Oppen (1980).

- $R \subseteq S \times S$ is a congruence relation if it is an equivalence relation
  - R is reflexive, symmetric, and transitive

  that satisfies for every n-ary function f the congruence condition of f:
  - $\forall t, u \in S^n.\ (\forall 1 \leq i \leq n.\ R(t_i, u_i)) \Rightarrow R(f(t), f(u))$
- The congruence closure R^c is the smallest congruence relation covering R:
  - R^c is a congruence relation with $R \subseteq R^c$
  - $\forall R'.\ (R'$ is a congruence relation with $R \subseteq R') \Rightarrow (R^c \subseteq R')$
- EUF-satisfiablity of formula $F :\Leftrightarrow (\bigwedge_{i=1}^{n} t_i = u_i) \wedge (\bigwedge_{j=n+1}^{n+m} t_j \neq u_j)$:
  - Let R be the relation $\{(t_i, u_i) \mid 1 \leq i \leq n\}$ on the set S of subterms of F.
  - F is EUF-satisfiable if and only if $\forall n+1 \leq j \leq n+m.\ \neg R^c(t_j, u_j)$.

To decide the EUF-satisfiability of F, it suffices to compute the congruence closure of the term equalities in F and check that it is compatible with the term inequalities.

Congruence Closure: Basic Idea

We compute the congruence closure by partitioning S into classes of congruent terms.

- Partition $S/R^c := \{[t]_{R^c} \mid t \in S\}$.
  - Congruence class $[t]_{R^c}$: $R^C(t,u)$ if and only if $[t]_{R^c} = [u]_{R^c}$.
  - Given F with equations $t_1 = u_1, \ldots, t_n = u_n$, compute partitions $P_0, P_1, \ldots, P_n = S/R^C$.
    - P_0: every element of S represents a separate congruence class.
    - P_{i+1}: determined from P_i by merging $[t_{i+1}]$ and $[u_{i+1}]$, i.e., by forming their union and propagating new congruences that arise within this union.
- Example: satisfiability of $F :\Leftrightarrow f(a,b) = a \land f(f(a,b),b) \neq a$
  - Set $S := \{a, b, f(a,b), f(f(a,b),b)\}$, single equation $f(a,b) = a$.
  - $P_0 := \{\{a\},\{b\},\{f(a,b)\},\{f(f(a,b),b)\}\}$
  - $P_1 := \{\{b\},\{a, f(a,b), f(f(a,b),b)\}\}$
    - Union of $[f(a,b)]$ and $[a]$: $\{\{b\},\{a, f(a,b)\},\{f(f(a,b),b)\}\}$
    - Propagation: $[f(a,b)] = [a]$ implies $[f(f(a,b),b)] = [f(a,b)]$
  - F is EUF-unsatisfiable: $[f(f(a,b),b)] = [a]$.

Congrence Closure: Algorithm

function CONGRUENCECLOSURE(S, R)
  $P := \{\{t\} \mid t \in S\}$ ▹ compute partition $P := S/(R^c)$
  for $(t, u) \in R$ **do**
    $P :=$ MERGE(S, P, t, u)
  end for ▹ return relation determined by P
  return $\{(t, u) \in S \times S \mid$ FIND$(P, t) =$ FIND$(P, u)\}$
end function

function CONGRUENT(P, t, u)
  if t and u are $f(t_1, \ldots, t_n)$ and $f(u_1, \ldots, u_n)$ **then**
    return $\forall 1 \leq i \leq n.$ FIND$(P, t_i) =$ FIND(P, u_i)
  else
    return false
  end if
end function

P can be represented by a "disjoint-set" data structure with efficient merge/find algorithms.

function MERGE(S, P, t, u) ▹ merge $[t]$ and $[u]$
  $p_t, p_u :=$ FIND(P, t), FIND(P, u)
  if $p_t = p_u$ **return** P
  $P := (P \setminus \{p_t, p_u\}) \cup \{p_t \cup p_u\}$
  for $(t_1, t_2) \in S \times S$ **do**
    $p_1, p_2 :=$ FIND(P, t_1), FIND(P, t_2)
    if $p_1 \neq p_2 \wedge$ CONGRUENT(P, t_1, t_2) **then**
      $P :=$ MERGE(P, p_1, p_2)
    end if
  end for
  return P
end function

function FIND(P, t) ▹ find congruence class $[t] \in P$
  choose $p \in P$ **with** $t \in p$
  return p
end function

Congruence Closure: More Examples

- Example: satisfiability of $F :\Leftrightarrow f(f(f(a))) = a \wedge f(f(f(f(f(a))))) = a \wedge f(a) \neq a$.
  - $P_0 := \{\{a\}, \{f(a)\}, \{f^2(a)\}, \{f^3(a)\}, \{f^4(a)\}, \{f^5(a)\}\}$
  - $P_1 := \{\{a, f^3(a)\}, \{f(a), f^4(a)\}, \{f^2(a), f^5(a)\}\}\}$
    - Union of $[f^3(a)]$ and $[a]$: $\{\{a, f^3(a)\}, \{f(a)\}, \{f^2(a)\}, \{f^4(a)\}, \{f^5(a)\}\}$
    - Propagation: $[f^3(a)] = [a]$ implies $[f^4(a)] = [f(a)]$ and $[f^5(a)] = [f^2(a)]$.
  - $P_2 := \{\{a, f(a), f^2(a), f^3(a), f^4(a), f^5(a)\}\}$
    - Union of $[f^5(a)]$ and $[a]$: $\{\{a, f^2(a), f^3(a), f^5(a)\}, \{f(a), f^4(a)\}\}$
    - Propagation: $[f^2(a)] = [a]$ implies $[f^3(a)] = [f(a)]$.
  - F is EUF-unsatisfiable: $[f(a)] = [a]$.
- Example: satisfiability of $F :\Leftrightarrow f(x) = y \wedge x \neq y$.
  - $P_0 := \{\{x\}, \{y\}, \{f(x)\}, \{f(y)\}\}$
  - $P_1 := \{\{x\}, \{y, f(x)\}, \{f(y)\}\}$
    - Union of $[f(x)]$ and $[y]$: $\{\{x\}, \{y, f(x)\}, \{f(y)\}\}$
    - No more propagation.
  - F is EUF-satisfiable: $[x] \neq [y]$.

The Theory E: Equality Logic

EUF without uninterpreted functions (i.e., only with constants).

- Decision of E-satisfiability:
  - Computation of congruence closure without the need to propagate congruences:

    function MERGE(S, P, t, u)
    $\quad p_t, p_u :=$ FIND(P, t), FIND(P, u)
    $\quad$ **return** $(P \setminus \{p_t, p_u\}) \cup \{p_t \cup p_u\}$ $\quad\quad$ ▹ equals P, if $p_t = p_u$
    end function

- Ackermann's Reduction: transformation of an EUF-formula into an E-formula.
  - Replace every function application $f(t_1, \ldots, t_n)$ by a fresh constant $f_{t_1,\ldots,t_n}$.
  - For every pair of applications $f(t_1, \ldots, t_n)$ and $f(u_1, \ldots, u_n)$, add the constraint

$$(t_1 = u_1 \wedge \ldots \wedge t_n = u_n) \Rightarrow f_{t_1,\ldots,t_n} = f_{u_1,\ldots,u_n}$$

  - The result is E-satisfiable if and only if the original formula is EUF-satisfiable.

The theory E needs larger formulas but has a simpler decision algorithm than EUF.

E-Satisfiability: Example

EUF-satisfiability of formula $F :\Leftrightarrow x_2 = x_3 \wedge f(x_1) = f(x_3) \wedge f(x_1) \neq f(x_2)$

- Ackermann's reduction to E-formula F':

$$x_2 = x_3 \wedge f_1 = f_3 \wedge f_1 \neq f_2 \wedge$$
$$(x_1 = x_2 \Rightarrow f_1 = f_2) \wedge (x_1 = x_3 \Rightarrow f_1 = f_3) \wedge (x_2 = x_3 \Rightarrow f_2 = f_3)$$

- Disjunctive normal form of F':

$$(\underline{x_2 = x_3} \wedge f_1 = f_3 \wedge f_1 \neq f_2 \wedge x_1 \neq x_2 \wedge x_1 \neq x_3 \wedge \underline{x_2 \neq x_3}) \vee$$
$$(x_2 = x_3 \wedge f_1 = f_3 \wedge f_1 \neq f_2 \wedge x_1 \neq x_2 \wedge x_1 \neq x_3 \wedge f_2 = f_3) \vee$$
$$(\underline{x_2 = x_3} \wedge f_1 = f_3 \wedge f_1 \neq f_2 \wedge x_1 \neq x_2 \wedge f_1 = f_3 \wedge \underline{x_2 \neq x_3}) \vee$$
$$(x_2 = x_3 \wedge f_1 = f_3 \wedge f_1 \neq f_2 \wedge x_1 \neq x_2 \wedge f_1 = f_3 \wedge f_2 = f_3) \vee$$
$$(x_2 = x_3 \wedge f_1 = f_3 \wedge \underline{f_1 \neq f_2} \wedge \underline{f_1 = f_2} \wedge x_1 \neq x_3 \wedge x_2 \neq x_3) \vee$$
$$(x_2 = x_3 \wedge f_1 = f_3 \wedge \underline{f_1 \neq f_2} \wedge \underline{f_1 = f_2} \wedge x_1 \neq x_3 \wedge f_2 = f_3) \vee$$
$$(x_2 = x_3 \wedge f_1 = f_3 \wedge \underline{f_1 \neq f_2} \wedge \underline{f_1 = f_2} \wedge f_1 = f_3 \wedge x_2 \neq x_3) \vee$$
$$(x_2 = x_3 \wedge f_1 = f_3 \wedge \underline{f_1 \neq f_2} \wedge \underline{f_1 = f_2} \wedge f_1 = f_3 \wedge f_2 = f_3)$$

E-Satisfiability: Example

E-satisfiability of DNF of F': only two clauses do not have conflicting literals.

- Satisfiability of $(x_2 = x_3 \land f_1 = f_3 \land f_1 \neq f_2 \land x_1 \neq x_2 \land x_1 \neq x_3 \land f_2 = f_3)$:
  - $P_0 := \{\{x_1\}, \{x_2\}, \{x_3\}, \{f_1\}, \{f_2\}, \{f_3\}\}$
  - $P_1 := \{\{x_1\}, \{x_2, x_3\}, \{f_1\}, \{f_2\}, \{f_3\}\}$
  - $P_2 := \{\{x_1\}, \{x_2, x_3\}, \{f_1, f_3\}, \{f_2\}\}$
  - $P_3 := \{\{x_1\}, \{x_2, x_3\}, \{f_1, f_2, f_3\}\}$
  - $[f_1] = [f_2]$: clause is E-unsatisfiable.
- Satisfiability of $(x_2 = x_3 \land f_1 = f_3 \land f_1 \neq f_2 \land x_1 \neq x_2 \land f_1 = f_3 \land f_2 = f_3)$:
  - $P_0 := \{\{x_1\}, \{x_2\}, \{x_3\}, \{f_1\}, \{f_2\}, \{f_3\}\}$
  - $P_1 := \{\{x_1\}, \{x_2, x_3\}, \{f_1\}, \{f_2\}, \{f_3\}\}$
  - $P_2 := \{\{x_1\}, \{x_2, x_3\}, \{f_1, f_3\}, \{f_2\}\}$
  - $P_3 := \{\{x_1\}, \{x_2, x_3\}, \{f_1, f_2, f_3\}\}$
  - $[f_1] = [f_2]$: clause is E-unsatisfiable.

DNF of F' is E-unsatisfiable, thus F is EUF-unsatisfiable.

Congruence Closure in OCaml

```
let rec subterms tm =
  match tm with
    Fn(f,args) -> itlist (union ** subterms) args [tm]
  | _ -> [tm];;
let congruent eqv (s,t) = (* Test whether subterms are congruent under an equivalence. *)
  match (s,t) with
    Fn(f,a1),Fn(g,a2) -> f = g & forall2 (equivalent eqv) a1 a2
  | _ -> false;;
let rec emerge (s,t) (eqv,pfn) = (* Merging of terms, with congruence closure. *)
  let s' = canonize eqv s and t' = canonize eqv t in
  if s' = t' then (eqv,pfn) else
  let sp = tryapplyl pfn s' and tp = tryapplyl pfn t' in
  let eqv' = equate (s,t) eqv in
  let st' = canonize eqv' s' in
  let pfn' = (st' |-> union sp tp) pfn in
  itlist (fun (u,v) (eqv,pfn) ->
              if congruent eqv (u,v) then emerge (u,v) (eqv,pfn)
              else eqv,pfn)
       (allpairs (fun u v -> (u,v)) sp tp) (eqv',pfn');;
```

EUF-Satisfiability/Validity in OCaml

```
let predecessors t pfn =
  match t with
    Fn(f,a) -> itlist (fun s f -> (s |-> insert t (tryapplyl f s)) f) (setify a) pfn
  | _ -> pfn;;
let ccsatisfiable fms = (* Satisfiability of conjunction of ground equations and inequations. *)
  let pos,neg = partition positive fms in
  let eqps = map dest_eq pos and eqns = map (dest_eq ** negate) neg in
  let lrs = map fst eqps @ map snd eqps @ map fst eqns @ map snd eqns in
  let pfn = itlist predecessors (unions(map subterms lrs)) undefined in
  let eqv,_ = itlist emerge eqps (unequal,pfn) in
  forall (fun (l,r) -> not(equivalent eqv l r)) eqns;;
let ccvalid fm = (* Validity checking a universal formula. *)
  let fms = simpdnf(askolemize(Not(generalize fm))) in
  not (exists ccsatisfiable fms);;

# ccvalid <<f(f(f(f(f(c))))) = c /\ f(f(f(c))) = c ==> f(c) = c \/ f(g(c)) = g(f(c))>>;;
- : bool = true
# ccvalid <<f(f(f(f(c)))) = c /\ f(f(c)) = c ==> f(c) = c>>;;
- : bool = true
```