Recent development in Tropical Differential Algebra

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References

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Motivation

From [1] the following is known.

Undecidability Result

There is no algorithm for computing formal power series solutions of systems of algebraic partial differential equations.

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There is no algorithm for computing formal power series solutions of systems of algebraic partial differential equations.

The goal of our work is to have a better understanding, and derive necessary conditions, of the support of solutions of systems of algebraic partial differential equations.

Overview

1. Algebraic Structures
   - Set of Vertices
   - Semiring of Vertex Sets
2. Tropicalization Map
3. Tropical Differential Algebra
   - Tropical Solution
4. Fundamental Theorem
5. Initials
   - Classical initial
   - Tropical Initials
   - Extended Fundamental Theorem
6. Recent Work

Set of Vertices

For $X \in \mathcal{P}(\mathbb{Z}_{\geq 0}^m)$ we define the Newton polytope $\mathcal{N}(X) \subseteq \mathbb{R}_{\geq 0}^m$ as the convex hull of

$$X + \mathbb{R}_{\geq 0}^m = \{x + (a_1, \ldots, a_m) \mid x \in X, a_1, \ldots, a_m \in \mathbb{R}_{\geq 0}\}.$$

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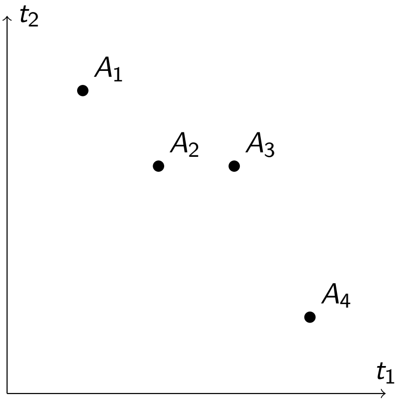
We call $x \in X$ a vertex if

$$x \notin \mathcal{N}(X \setminus \{x\}),$$

and we denote by $\operatorname{Vert}(X)$ the set of vertices of X.

Example 1

Let $X = \{A_1 = (1,4), A_2 = (2,3), A_3 = (3,3), A_4 = (4,1)\} \subseteq \mathbb{Z}^2_{\geq 0}$.



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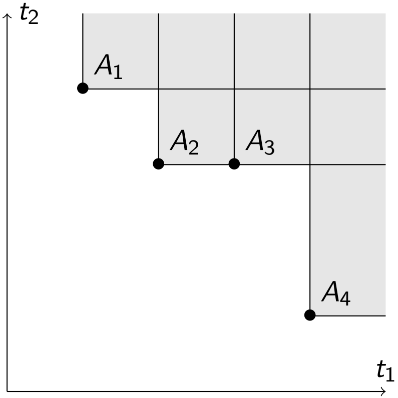
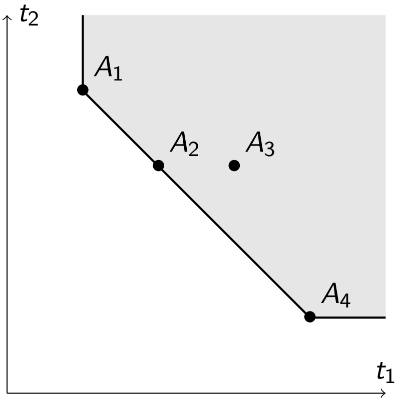


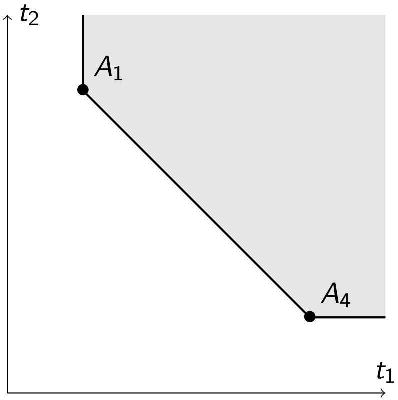
Example 1

The Newton polytope $\mathcal{N}(X)$ looks as follows.



Example 1

The set of vertices is $\text{Vert}(X) = \{A_1, A_4\}$.



Set of Vertices

Lemma

Let $X, Y \in \mathcal{P}(\mathbb{Z}^m_{\geq 0})$*. Then*

- $\mathcal{N}(\mathrm{Vert}(X)) = \mathcal{N}(X)$.

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- $\mathcal{N}(\mathrm{Vert}(X)) = \mathcal{N}(X)$.

Moreover, The following statements are equivalent:

- $\mathrm{Vert}(X) = \mathrm{Vert}(Y)$;
- $\mathcal{N}(X) = \mathcal{N}(Y)$;
- *There is* $Z \in \mathcal{P}(\mathbb{Z}^m_{\geq 0})$ *such that* $X + Z = Y + Z$.

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Let $X, Y \in \mathcal{P}(\mathbb{Z}^m_{\geq 0})$. Then

- $\mathcal{N}(\mathrm{Vert}(X)) = \mathcal{N}(X)$.

Moreover, The following statements are equivalent:

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- $\mathcal{N}(X) = \mathcal{N}(Y)$;
- *There is $Z \in \mathcal{P}(\mathbb{Z}^m_{\geq 0})$ such that $X + Z = Y + Z$.*

As a consequence, $\mathrm{Vert}(X)$ is the least set generating $\mathcal{N}(X)$ (with respect to " $\subseteq$ " as ordering).

Vertex Map

With abuse of notation we define the map

$$\mathrm{Vert}\colon \mathcal{P}(\mathbb{Z}_{\geq 0}^{m}) \longrightarrow \mathcal{P}(\mathbb{Z}_{\geq 0}^{m}),$$

where X is projected onto its set of vertices $\mathrm{Vert}(X)$.

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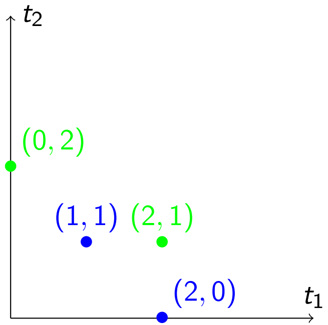
where X is projected onto its set of vertices $\mathrm{Vert}(X)$.
We denote by $\mathbb{T}_m$ the image of Vert and define for $X, Y \in \mathbb{T}_m$ the operations

- $X \oplus Y = \mathrm{Vert}(X \cup Y)$;
- $X \odot Y = \mathrm{Vert}(X + Y)$.

Example 2

Let us consider the vertex sets

$$X = \{(2,0),(1,1)\},\ Y = \{(0,2),(2,1)\}.$$



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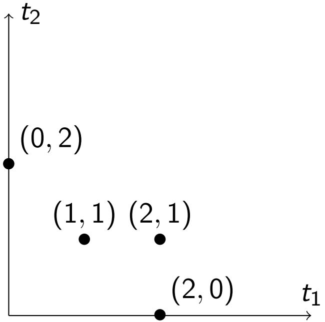

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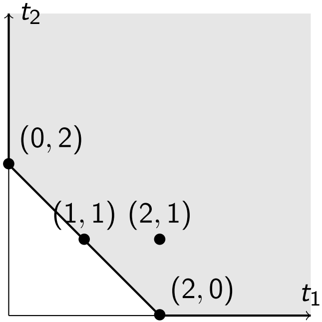

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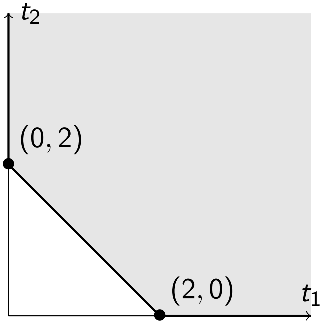

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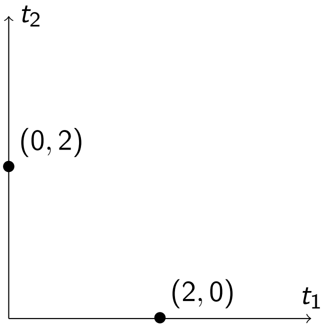

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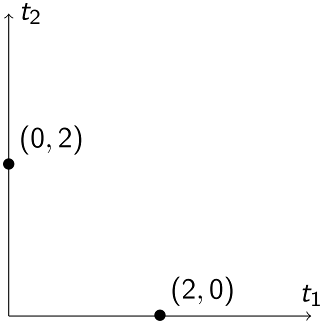

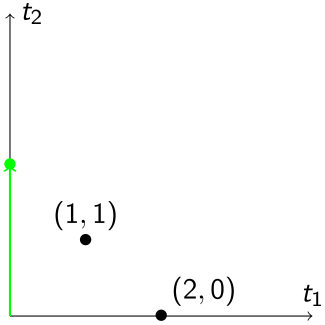

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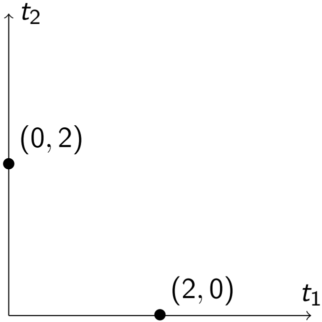

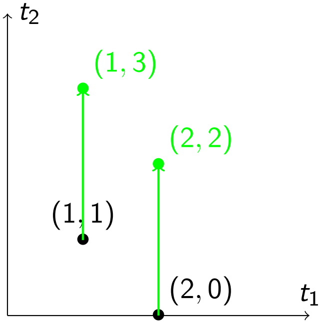

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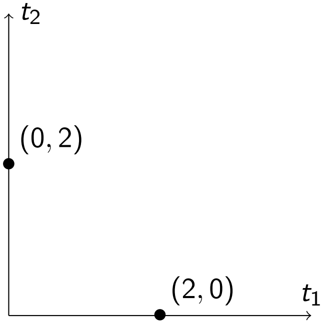

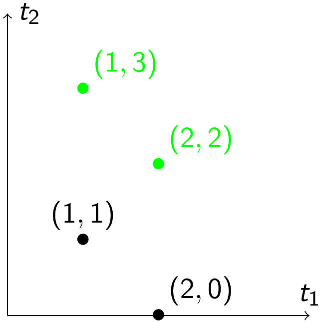

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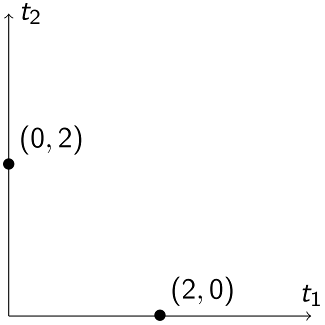

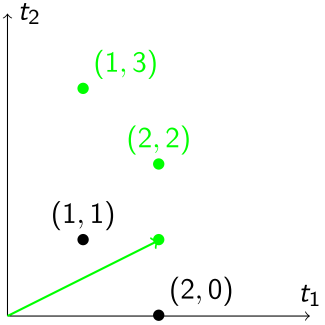

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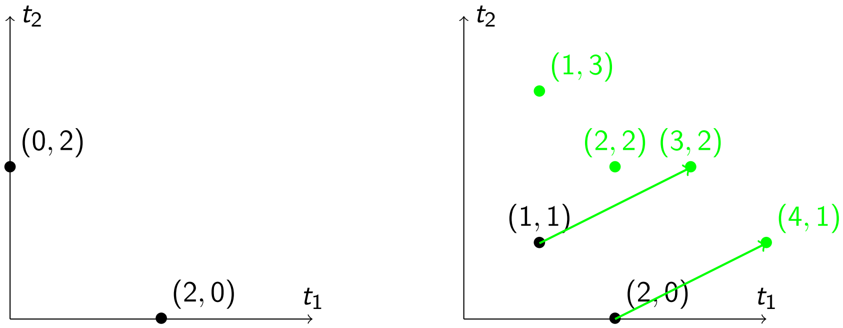

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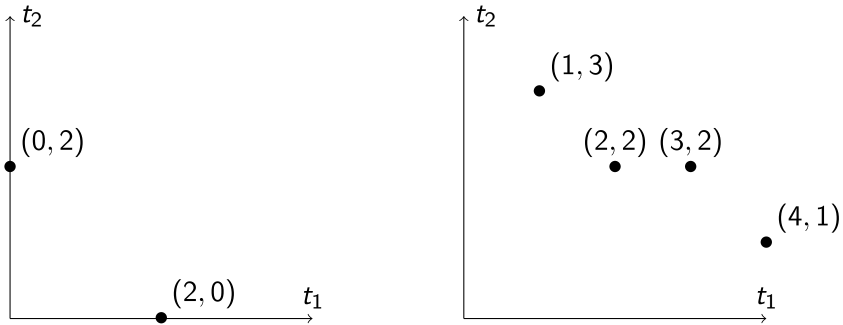

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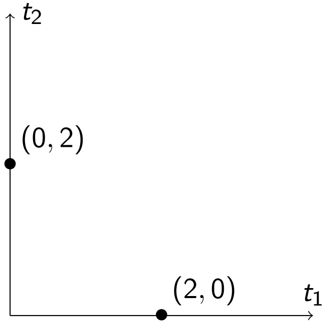

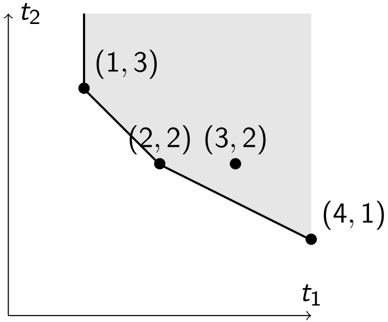

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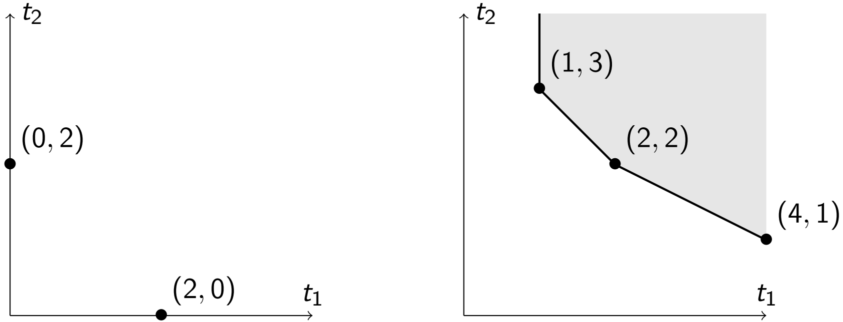

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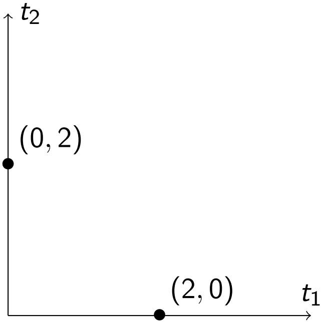

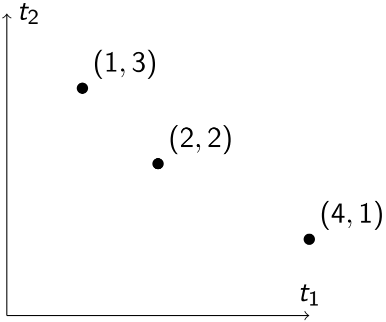

Semiring of Vertex Sets

Lemma

$(\mathbb{T}_m, \oplus, \odot, \emptyset, \{0, \dots, 0\})$ *is a commutative idempotent semiring, i.e. for all* $a, b, c \in \mathbb{T}_m$

- $(\mathbb{T}_m, \oplus, \emptyset), (\mathbb{T}_m, \odot, \{0, \dots, 0\})$ *are commutative monoids;*
- $a \odot (b \oplus c) = a \odot b \oplus a \odot c$*;*
- $\emptyset \odot a = \emptyset$*;*
- $a \oplus a = a$*.*

Tropicalization Map

Let K be an algebraically closed field of characteristic zero and $m \geq 1$. The support of $\varphi = \sum a_J t^J \in K[[t_1, \dots, t_m]]$ is defined as

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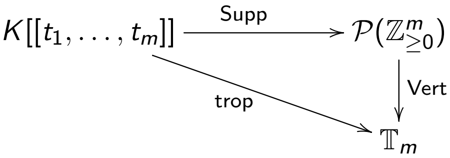
The tropicalization map is defined as

$$\begin{array}{rccc} \mathrm{trop}\colon & K[[t_1, \ldots, t_m]] & \to & \mathbb{T}_m \\ & \varphi & \mapsto & \mathrm{Vert}(\mathrm{Supp}(\varphi)) \end{array}$$



Tropicalization Map

Lemma

The tropicalization map is a non-degenerate valuation, i.e. for all $\varphi, \psi \in K[[t_1, \ldots, t_m]]$

- $\mathrm{trop}(0) = \emptyset$, $\mathrm{trop}(\pm 1) = \{(0, \ldots, 0)\}$;
- $\mathrm{trop}(\varphi \cdot \psi) = \mathrm{trop}(\varphi) \odot \mathrm{trop}(\psi)$;
- $\mathrm{trop}(\varphi + \psi) \oplus \mathrm{trop}(\varphi) \oplus \mathrm{trop}(\psi) = \mathrm{trop}(\varphi) \oplus \mathrm{trop}(\psi)$;
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These properties are the essence in the proof of the Fundamental Theorem and one of the difficulties of the paper [3] was to find a “good” definition of the map trop which satisfies them.

Differential Polynomials

For $J = (j_1, \ldots, j_m) \in \mathbb{Z}^m_{\geq 0}$ we denote by $\Theta(J)$ the differential operator

$$\Theta(J) = \frac{\partial^{j_1+\cdots+j_m}}{\partial t_1^{j_1} \cdots \partial t_m^{j_m}},$$

where $\frac{\partial}{\partial t_k}$ is the partial derivative with respect to t_k.

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where $\frac{\partial}{\partial t_k}$ is the partial derivative with respect to t_k. Then for $\varphi \in K[[t_1, \dots, t_m]]$ we obtain

$$\mathrm{Supp}(\Theta(J)\varphi) = \left\{ (s_1 - j_1, \dots, s_m - j_m) \;\middle|\; \begin{array}{c} (s_1, \dots, s_m) \in \mathrm{Supp}(\varphi), \\ s_i - j_i \geq 0 \text{ for all } i \end{array} \right\}.$$

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For $\varphi = t_1^2 + t_1 t_2 - t_2$ we have $\mathrm{Supp}(\varphi) = \{(2,0), (1,1), (0,1)\}$ and

$$\mathrm{Supp}(\Theta(1,0)\varphi) = \{(1,0), (0,1)\}.$$

Differential Polynomials

A differential monomial of order $r \in \mathbb{Z}_{\geq 0}$ depending on differential indeterminates $x_1, \ldots, x_n$ can be written as

$$E_M = \prod_{\substack{1 \leq i \leq n \\ \max(J) \leq r}} (\Theta(J) x_i)^{M_{i,J}}$$

for some $M = (M_{i,J}) \in (\mathbb{Z}_{\geq 0})^{n \times (r+1)^m}$.

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for some $M = (M_{i,J}) \in (\mathbb{Z}_{\geq 0})^{n \times (r+1)^m}$.
A differential polynomial is an expression of the form

$$P = \sum_M \alpha_M \cdot E_M,$$

where finitely many coefficients $\alpha_M \in K[[t_1, \ldots, t_m]]$ are non-zero and E_M are differential monomials.

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A set $\Sigma \subseteq K[[t_1, \dots, t_m]]\{x_1, \dots, x_n\}$ is called a differential ideal if

- Σ is an ideal of $K[[t_1, \dots, t_m]]\{x_1, \dots, x_n\}$;
- For every $P \in \Sigma, J \in \mathbb{Z}^m_{\geq 0}$ it holds that

$$\Theta(J)P \in \Sigma.$$

Tropical Derivative Operator

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A tropical derivative operator $\Theta_{\text{trop}}(J) \colon \mathcal{P}(\mathbb{Z}_{\geq 0}^m) \to \mathcal{P}(\mathbb{Z}_{\geq 0}^m)$ is defined as

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For $S = \operatorname{Supp}(t_1^2 + t_1 t_2 - t_2) = \{(2,0),(1,1),(0,1)\}$ we have

$$\Theta_{\text{trop}}(1,0)S = \{(1,0),(0,1)\}.$$

Tropical Differential Polynomial

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$$E = \prod_{\substack{1 \le i \le n \\ \max(J) \le r}} (\Theta(J)x_i)^{M_{i,J}} \quad \longleftrightarrow \quad E_{\text{trop}} = \bigodot_{\substack{1 \le i \le n \\ \max(J) \le r}} \text{Vert}(\Theta_{\text{trop}}(J)S_i)^{\odot M_{i,J}}$$

Tropical Differential Polynomial

By applying the tropicalization map, we obtain the corresponding tropical differential monomial and tropical differential polynomial, respectively.

$$E = \prod_{\substack{1 \le i \le n \\ \max(J) \le r}} (\Theta(J) x_i)^{M_{i,J}} \quad \longleftrightarrow \quad E_{\text{trop}} = \bigodot_{\substack{1 \le i \le n \\ \max(J) \le r}} \text{Vert}(\Theta_{\text{trop}}(J) S_i)^{\odot M_{i,J}}$$

$$P = \sum_M \alpha_M \cdot E_M \qquad \longleftrightarrow \qquad P_{\text{trop}} = \bigoplus_M \text{trop}(\alpha_M) \odot E_{M,\text{trop}}$$

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$$P = t_1(\Theta(1,0)x_1)^2 + 2t_2^2 x_2 \quad \longleftrightarrow \quad P_{\text{trop}} = \{(1,0)\} \odot (\Theta_{\text{trop}}(1,0)S_1)^{\odot 2} \oplus \{(0,2)\} \odot S_2$$

Tropical Solution

Let

$$P_{\text{trop}} = \bigoplus_{M \in \Delta} a_M \odot \epsilon_M$$

be a tropical differential polynomial. An n-tuple $S \in \mathcal{P}(\mathbb{Z}_{\geq 0}^m)^n$ is called a solution of P_{trop} if for every $J \in P_{\text{trop}}(S)$ there exist $M_1, M_2 \in \Delta$ with $M_1 \neq M_2$ such that

$$J \in a_{M_1} \odot \epsilon_{M_1}(S) \quad \text{and} \quad J \in a_{M_2} \odot \epsilon_{M_2}(S).$$

Fundamental Theorem

Goal

We now want to find a relation between the solutions of the original system of differential equations and the solutions of the corresponding tropical differential polynomials.

Example 3

Let

$$P = t \cdot \frac{\partial x}{\partial t} - x.$$

The solutions of $P = 0$ are $\varphi = c\,t$, where $c \in K$. Hence, $\mathrm{Supp}(\varphi) = \{1\}$ or $\mathrm{Supp}(\varphi) = \emptyset$, respectively, and

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The corresponding tropical differential polynomial is

$$\begin{aligned} P_{\text{trop}}(S) &= \{(1)\} \odot \Theta_{\text{trop}}(1)S \oplus S \\ &= \text{Vert}(\text{Vert}(\{1\} + \Theta_{\text{trop}}(1)S) \cup \text{Vert}(S))). \end{aligned}$$

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Let S be a solution with $0 \in S$. Then $0 \in \mathrm{Vert}(S)$ and $0 \in P_{\mathrm{trop}}(S)$. But $0 \notin \mathrm{Vert}(\{1\} + \Theta_{\mathrm{trop}}(1)S)$ in contradiction to the assumption that S is a solution.

Example 3

Since $0 \notin S$, it holds that

$$\mathrm{Vert}(\{1\} + \Theta_{\mathrm{trop}}(1)S) = \mathrm{Vert}(S)$$

and for every $J \in P_{\mathrm{trop}}(S)$ with $J \in \mathbb{Z}_{>0}$ we obtain $J \in \mathrm{Vert}(S)$ and $J \in \mathrm{Vert}(\{1\} + \Theta_{\mathrm{trop}}(1)S)$.

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This means that we do not obtain more conditions on the support of the solutions of $P = 0$ by considering P_{trop}.

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The case $1 \in (\Theta(1)P)_{\text{trop}}(S)$ corresponds to $S = \{2, \mathcal{O}(3)\}$. But 1 corresponds to a single tropical monomial and hence, S is not a solution.
Similarly, for $J \in S$ with $J \geq 2$ we obtain that J is the vertex of only one tropical monomial and S cannot be a solution.
Only $(\Theta(1)P)_{\text{trop}}(S) = \emptyset$ remains which leads to $S = \emptyset$ or $S = \{1\}$ and hence,

$$\text{Sol}(P_{\text{trop}}, (\Theta(1)P)_{\text{trop}}\} = \{\emptyset, \{1\}\}.$$

Example 3

To summarize, we have obtained that

$$\text{Supp}(\text{Sol}(P, \Theta(1)P)) = \text{Sol}(P_{\text{trop}}, (\Theta(1)P)_{\text{trop}}).$$

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$$\mathrm{Supp}(\mathrm{Sol}(P, \Theta(1)P)) = \mathrm{Sol}(P_{\mathrm{trop}}, (\Theta(1)P)_{\mathrm{trop}}).$$

We now want to precisely state this observation as the Fundamental Theorem.

Fundamental Theorem

Fundamental Theorem

Let K be an uncountable, algebraically closed field of characteristic zero. Let Σ be a differential ideal in the ring $K[[t_1, \ldots, t_m]]\{x_1, \ldots, x_n\}$. Then

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" $\subseteq$ " : holds for more general K.
" $\supseteq$ " : uses ultrapower construction similar to the proof of the Strong Approximation Theorem in [1].

Initial of Differential Polynomials (Ritt, Kolchin, etc.)

An orderly ranking is a total order on the set of differential operators $\{\Theta(J) \mid J \in \mathbb{Z}_{\geq 0}^m\}$ which regards derivatives, i.e. for all $I, J, M \in \mathbb{Z}_{\geq 0}^m$

- $\Theta(J) \leq \Theta(I)\Theta(J)$;
- $\Theta(J) \leq \Theta(I) \implies \Theta(M+J) \leq \Theta(M+I)$.

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For

$$P = t_1\Theta(1,1)x_1(\Theta(2,0)x_2)^3 - t_1^3(\Theta(2,0)x_1)^2 + t_2^2(\Theta(0,1)x_2)^2$$

and an orderly ranking with $\Theta(0,1) < \Theta(1,1) < \Theta(2,0)$ we obtain

$$init(P) = t_1\Theta(1,1)x_1.$$

Motivation for initials

The initial (and separant) of a differential polynomial contain the main information for simplifying (systems of) equations and finding its solutions, for example the computation of differential regular chains, Thomas decomposition and Cauchy-Kovalevski like algorithms.

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Goal

We want to find an initial of the differential polynomial containing the main information with respect to tropicalization and solving the tropical version.

For $a \in K[[t_1, \ldots, t_m]]$ we denote by $\overline{a}$ the restriction of a to its set of vertices.

For $a = t_1^2 + t_1 t_2 - 2t_2^2$ we have $\overline{a} = t_1^2 - 2t_2^2$.

Tropical Initials

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Let $P = \sum_{M \in \Lambda} a_M E_M$ and $S \in \mathcal{P}(\mathbb{Z}_{\geq 0}^m)^n$. Then we define the initial of P (with respect to S) as

$$init_S(P) = \sum_{\substack{M \in \Lambda \\ \text{trop}(a_M E_M)(S) \cap p(S) \neq \emptyset}} \overline{a_M} E_M.$$

Tropical Initials

Example 4

Let $P = x_{(1,0)} + x_{(0,1)}$ and $\varphi = \alpha\, t_1^2 + \beta\, t_2^2$. Then φ is not a solution of $P = 0$ for any $\alpha, \beta \neq 0$, but we obtain $init_S(P) = P$ for $S = \mathrm{Supp}(\varphi) = \{(2,0),(0,2)\}$.

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For $\Theta(1,0)P$, however, we obtain $\Theta(1,0)P = x_{(2,0)} + x_{(1,1)}$ and

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If we find a single monomial in the tropical initial, this cannot have any solution.

Question

Does the converse direction hold as well?

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For P itself the answer is NO (see Example 4).

Initial Ideal

Instead we consider the initials of the differential ideal generated by P and from that its algebraic ideal:

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Let $\Sigma \subset K[[t_1, \ldots, t_m]]\{x_1, \ldots, x_n\}$ be a differential ideal and $S \in \mathcal{P}(\mathbb{Z}^m_{\geq 0})^n$. Then we define the initial ideal $init_S(\Sigma)$ with respect to S as the algebraic ideal generated by

$$\{init_S(P) \,:\, P \in G\} \subset K[[t_1, \ldots, t_m]]\{x_1, \ldots, x_n\}.$$

Initial ideal

Taking derivatives and considering the initial does NOT commute. This is why we have to choose the smaller set of the algebraic ideal (instead of the differential ideal) for $init_S(\Sigma)$.

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Let $P = x_{(0)} - 1$ and $S = \{0\}$. Then $init_S(P) = P$ and for every J it holds that $init_S(\Theta(J)P) = 0$. Hence,

$$init_S([P]) = \langle P \rangle.$$

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But $\Theta(1)P \in [init_S(P)]$ and the differential ideal generated by the initials is not monomial-free. In fact, $\varphi = 1$ is a solution with $S = \mathrm{Supp}(\varphi)$.

Extended Fundamental Theorem

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Let K be an uncountable, algebraically closed field of characteristic zero. Let Σ be a differential ideal in the ring $K[[t_1, \ldots, t_m]]\{x_1, \ldots, x_n\}$. Then the following three subsets of $\left(\mathcal{P}(\mathbb{Z}^m_{\geq 0})\right)^n$ coincide:

1. $\mathrm{Supp}(\mathrm{Sol}(\Sigma))$,
2. $\mathrm{Sol}(\Sigma_{\mathrm{trop}})$,
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" $\Rightarrow$ " : careful analysis and rewriting of the restricted coefficients and the corresponding set of vertices.

" $\Leftarrow$ " : can be carried out from the proof of the Fundamental Theorem.

Recent Work

The notion of initial degeneration should be properly defined for differential ideals in $K[[t_1, \ldots, t_m]]\{x_1, \ldots, x_n\}$.

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The notion of initial degeneration should be properly defined for differential ideals in $K[[t_1, \ldots, t_m]]\{x_1, \ldots, x_n\}$. The steps in this direction are

1. Consider the underlying structures $K((t_1, \ldots, t_m))$ and $Frac(\mathbb{T}_m)$. Of particular interest are prime ideals in

$$K((t_1, \ldots, t_m))^\circ = \{\frac{\varphi}{\psi} : \mathcal{N}(\text{trop}(\varphi)) \subseteq \mathcal{N}(\text{trop}(\psi))\}.$$

For example, $K((t_1, \ldots, t_m))^\circ$ is a non-Noetherian Bézout domain.

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3. Consider the (differential) varieties
$$X = \text{Spec}(K((t_1,\ldots,t_m))\{x_1,\ldots,x_n\}/\Sigma),$$
defined over $K((t_1,\ldots,t_m))$, where $\Sigma \subset K((t_1,\ldots,t_m))\{x_1,\ldots,x_n\}$ is a differential ideal. Then we are interested into the affine schemes
$$\chi(S) = \text{Spec}(K((t_1,\ldots,t_m))^\circ\{x_1,\ldots,x_n\}/\Sigma_S),$$
where Σ_S are the translated differential ideals and its fibers $\chi(S)_\nu$.

Recent Work

Additionally, we want to find the right notion of tropical bases for differential ideals in $K((t_1, \ldots, t_m))\{x_1, \ldots, x_n\}$. In particular, our goal is to

1. reproduce the initial ideal;
2. generalize differential Gröbner bases;
3. obtain a reduction process for differential polynomials.