The Fundamental Theorem of Tropical Partial Differential Algebraic Geometry

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References

F. Aroca, C. Garay, Z. Toghani, *The Fundamental Theorem of Tropical Differential Algebraic Geometry*. Pacific Journal of Mathematics, 283(2):257–270, 2016.

J. Denef, L. Lipshitz, *Power Series Solutions of Algebraic Differential Equations*. Mathematische Annalen, 267:213–238, 1984.

Motivation

From [2] the following is known.

Undecidability Result

There is no algorithm for computing formal power series solutions of systems of algebraic partial differential equations.

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The goal of this work is to have a better understanding, and derive necessary conditions, of the support of solutions of systems of algebraic partial differential equations.

Overview

1. Algebraic Structures
   - Set of Vertices
   - Semiring of Vertex Sets

2. Tropicalization Map

3. Tropical Differential Algebra
   - Tropical Solution

4. Fundamental Theorem

Set of Vertices

For $X \in \mathcal{P}(\mathbb{Z}_{\geq 0}^m)$ we define the Newton polytope $\mathcal{N}(X) \subseteq \mathbb{R}_{\geq 0}^m$ as the convex hull of

$$X + \mathbb{R}_{\geq 0}^m = \{x + (a_1, \ldots, a_m) \mid x \in X, a_1, \ldots, a_m \in \mathbb{R}_{\geq 0}\}.$$

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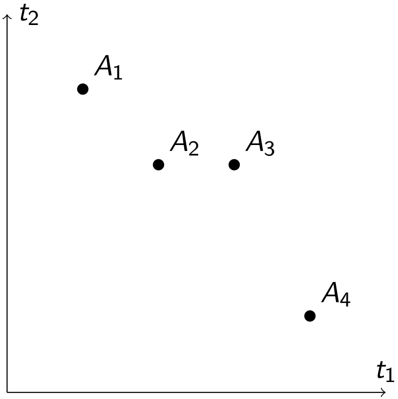
We call $x \in X$ a vertex if

$$x \notin \mathcal{N}(X \setminus \{x\}),$$

and we denote by $\mathrm{Vert}(X)$ the set of vertices of X.

Example 1

Let $X = \{A_1 = (1,4), A_2 = (2,3), A_3 = (3,3), A_4 = (4,1)\} \subseteq \mathbb{Z}^2_{\geq 0}$.



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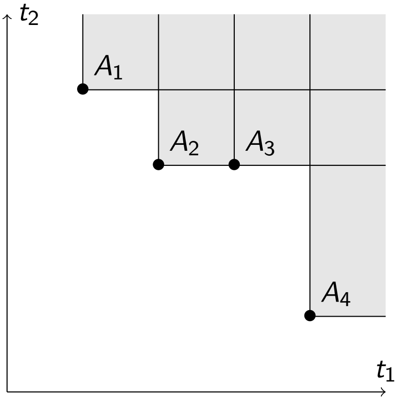
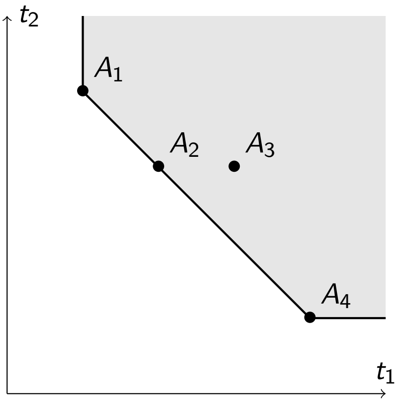


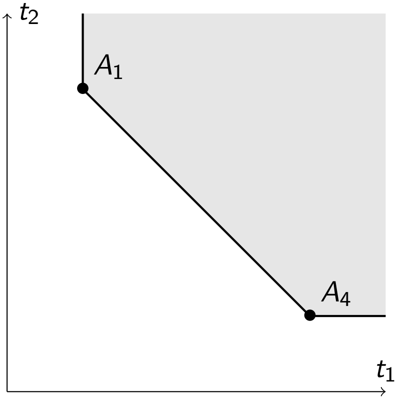
Example 1

The Newton polytope $\mathcal{N}(X)$ looks as follows.



Example 1

The set of vertices is $\mathrm{Vert}(X) = \{A_1, A_4\}$.



Set of Vertices

Lemma

Let $X, Y \in \mathcal{P}(\mathbb{Z}^m_{\geq 0})$*. Then*

- $\mathcal{N}(\mathrm{Vert}(X)) = \mathcal{N}(X)$*;*
- $\mathrm{Vert}(X) = \mathrm{Vert}(Y)$ *if and only if* $\mathcal{N}(X) = \mathcal{N}(Y)$.

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- $\mathrm{Vert}(X) = \mathrm{Vert}(Y)$ *if and only if* $\mathcal{N}(X) = \mathcal{N}(Y)$.

As a consequence, $\mathrm{Vert}(X)$ is the least set generating $\mathcal{N}(X)$ (with respect to " $\subseteq$ " as ordering).

Vertex Map

With abuse of notation we define the map

$$\mathrm{Vert}\colon \mathcal{P}(\mathbb{Z}^m_{\geq 0}) \longrightarrow \mathcal{P}(\mathbb{Z}^m_{\geq 0}),$$

where X is projected onto its set of vertices $\mathrm{Vert}(X)$.

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$$\text{Vert}\colon \mathcal{P}(\mathbb{Z}^m_{\geq 0}) \longrightarrow \mathcal{P}(\mathbb{Z}^m_{\geq 0}),$$

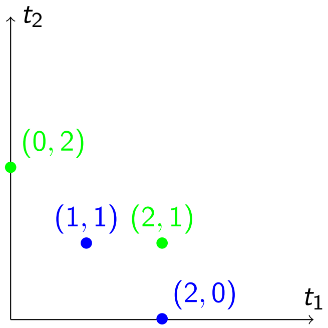
where X is projected onto its set of vertices $\text{Vert}(X)$.
We denote by $\mathbb{T}_m$ the image of Vert and define for $X, Y \in \mathbb{T}_m$ the operations

- $X \oplus Y = \text{Vert}(X \cup Y)$;
- $X \odot Y = \text{Vert}(X + Y)$.

Example 2

Let us consider the vertex sets

$$X = \{(2,0),(1,1)\},\ Y = \{(0,2),(2,1)\}.$$



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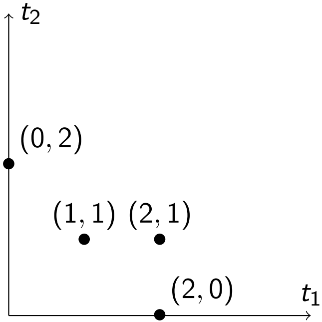

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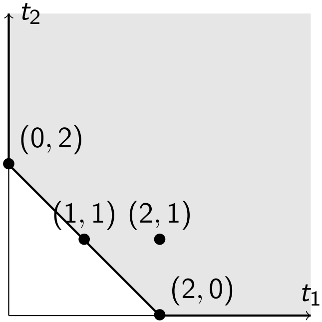

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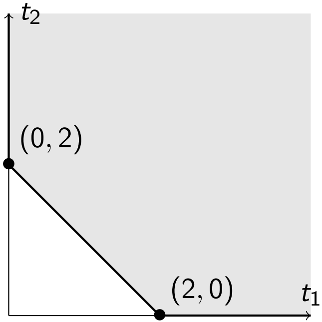

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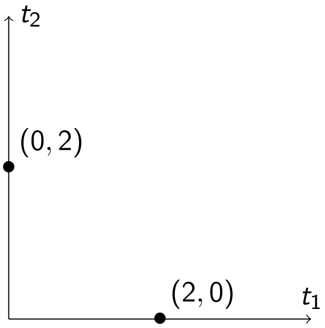

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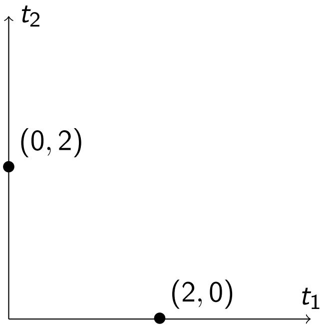

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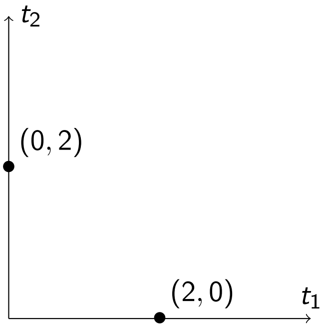

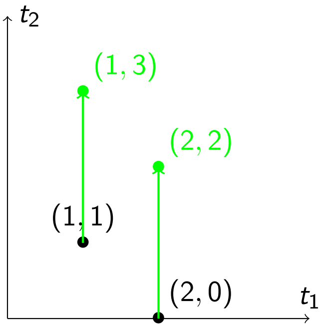

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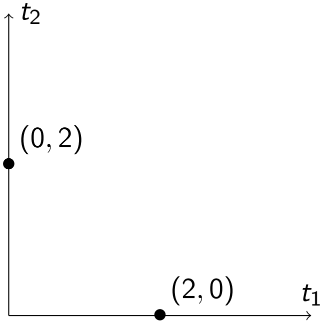

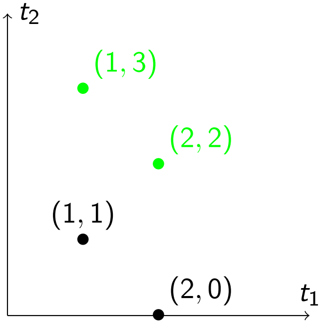

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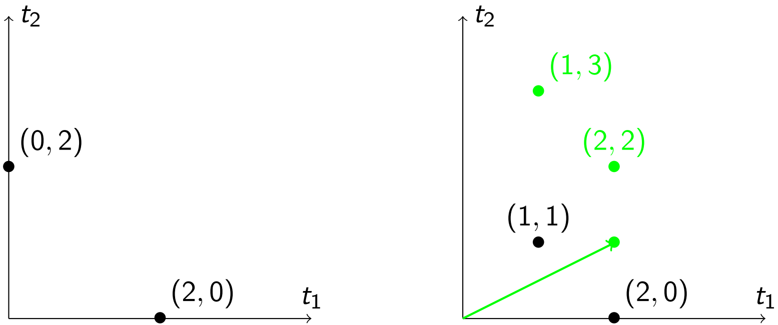

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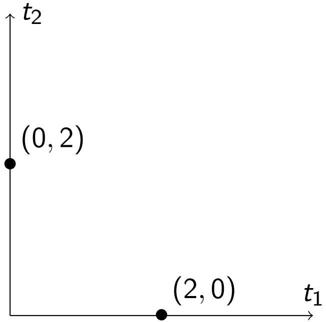

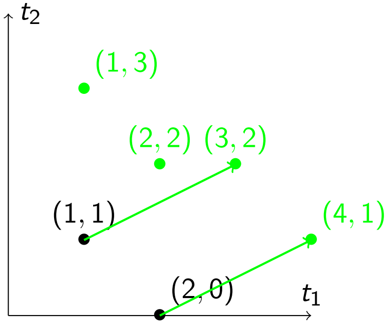

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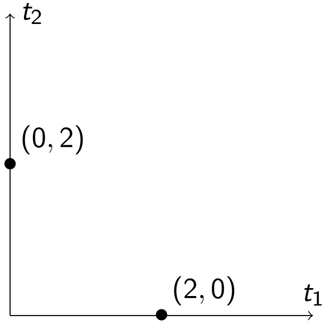

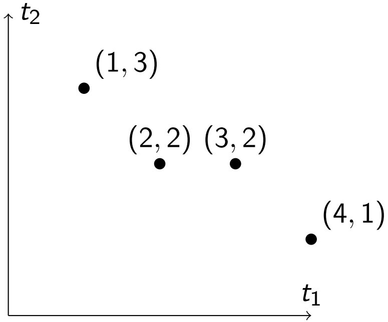

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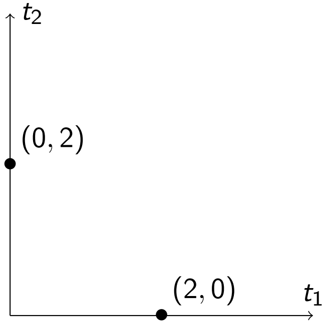

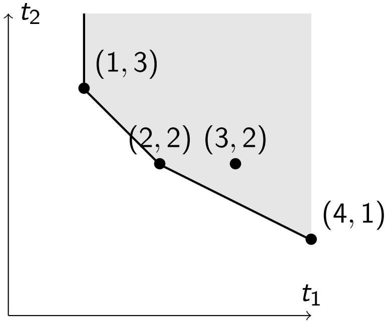

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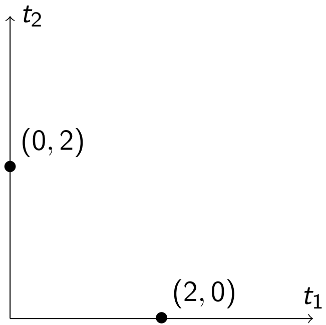

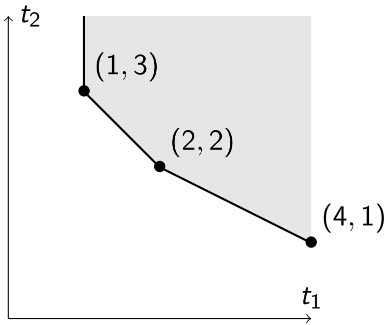

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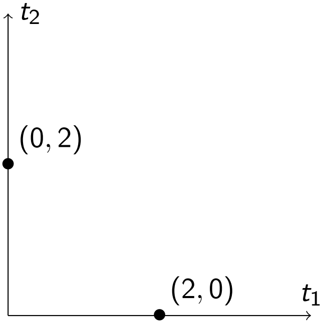

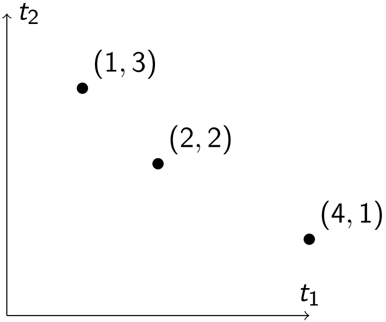

Semiring of Vertex Sets

Lemma

$(\mathbb{T}_m, \oplus, \odot, \emptyset, \{0, \ldots, 0\})$ *is a commutative idempotent semiring, i.e. for all* $a, b, c \in \mathbb{T}_m$

- $(\mathbb{T}_m, \oplus, \emptyset), (\mathbb{T}_m, \odot, \{0, \ldots, 0\})$ *are commutative monoids;*
- $a \odot (b \oplus c) = a \odot b \oplus a \odot c$*;*
- $\emptyset \odot a = \emptyset$*;*
- $a \oplus a = a$*.*

Tropicalization Map

Let K be an algebraically closed field of characteristic zero and $m \geq 1$. The support of $\varphi = \sum a_J t^J \in K[[t_1, \ldots, t_m]]$ is defined as

$$\mathrm{Supp}(\varphi) = \{J \in \mathbb{Z}_{\geq 0}^m \mid a_J \neq 0\}.$$

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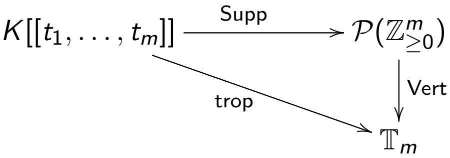
The tropicalization map is defined as

$$\begin{array}{rccc} \mathrm{trop}\colon & K[[t_1, \ldots, t_m]] & \to & \mathbb{T}_m \\ & \varphi & \mapsto & \mathrm{Vert}(\mathrm{Supp}(\varphi)) \end{array}$$



Tropicalization Map

Lemma

The tropicalization map is a non-degenerate valuation, i.e. for all $\varphi, \psi \in K[[t_1, \ldots, t_m]]$

- $\mathrm{trop}(0) = \emptyset$, $\mathrm{trop}(\pm 1) = \{(0, \ldots, 0)\}$;
- $\mathrm{trop}(\varphi \cdot \psi) = \mathrm{trop}(\varphi) \odot \mathrm{trop}(\psi)$;
- $\mathrm{trop}(\varphi + \psi) \oplus \mathrm{trop}(\varphi) \oplus \mathrm{trop}(\psi) = \mathrm{trop}(\varphi) \oplus \mathrm{trop}(\psi)$;
- $\mathrm{trop}(\varphi) = \emptyset$ *implies that* $\varphi = 0$.

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These properties are the essence in the proof of the main theorem and one of the difficulties of this paper was to find a "good" definition of the map trop which satisfies them.

Differential Polynomials

For $J = (j_1, \ldots, j_m) \in \mathbb{Z}^m_{\geq 0}$ we denote by $\Theta(J)$ the differential operator

$$\Theta(J) = \frac{\partial^{j_1 + \cdots + j_m}}{\partial t_1^{j_1} \cdots \partial t_m^{j_m}},$$

where $\frac{\partial}{\partial t_k}$ is the partial derivative with respect to t_k.

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where $\frac{\partial}{\partial t_k}$ is the partial derivative with respect to t_k. Then for $\varphi \in K[[t_1, \ldots, t_m]]$ we obtain

$$\mathrm{Supp}(\Theta(J)\varphi) = \left\{ (s_1 - j_1, \ldots, s_m - j_m) \;\middle|\; \begin{array}{c} (s_1, \ldots, s_m) \in \mathrm{Supp}(\varphi), \\ s_i - j_i \geq 0 \text{ for all } i \end{array} \right\}.$$

Differential Polynomials

A differential monomial of order $r \in \mathbb{Z}_{\geq 0}$ depending on differential indeterminates $x_1, \ldots, x_n$ can be written as

$$E_M = \prod_{\substack{1 \leq i \leq n \\ \max(J) \leq r}} (\Theta(J) x_i)^{M_{i,J}}$$

for some $M = (M_{i,J}) \in (\mathbb{Z}_{\geq 0})^{n \times (r+1)^m}$.

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for some $M = (M_{i,J}) \in (\mathbb{Z}_{\geq 0})^{n \times (r+1)^m}$.
A differential polynomial is an expression of the form

$$P = \sum_M \alpha_M \cdot E_M,$$

where finitely many coefficients $\alpha_M \in K[[t_1, \ldots, t_m]]$ are non-zero and E_M are differential monomials.

Differential Ideals

The ring consisting of all differential polynomials in the variables $x_1, \dots, x_n$ will be denoted by

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A set $\Sigma \subseteq K[[t_1, \ldots, t_m]]\{x_1, \ldots, x_n\}$ is called a differential ideal if

- Σ is an ideal of $K[[t_1, \ldots, t_m]]\{x_1, \ldots, x_n\}$;
- For every $P \in \Sigma, J \in \mathbb{Z}_{\geq 0}^m$ it holds that

$$\Theta(J)P \in \Sigma.$$

Tropical Derivative Operator

Let us define the corresponding tropical operations and object.

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A tropical derivative operator $\Theta_{\text{trop}}(J) \colon \mathcal{P}(\mathbb{Z}^m_{\geq 0}) \to \mathcal{P}(\mathbb{Z}^m_{\geq 0})$ is defined as

$$\Theta_{\text{trop}}(J)S = \left\{ (s_1 - j_1, \ldots, s_m - j_m) \;\middle|\; \begin{array}{c} (s_1, \ldots, s_m) \in S, \\ s_i - j_i \geq 0 \text{ for all } i \end{array} \right\}.$$

Tropical Differential Polynomial

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$$E = \prod_{\substack{1 \leq i \leq n \\ \max(J) \leq r}} (\Theta(J) x_i)^{M_{i,J}} \quad \longleftrightarrow \quad E_{\mathrm{trop}} = \bigodot_{\substack{1 \leq i \leq n \\ \max(J) \leq r}} \mathrm{Vert}(\Theta_{\mathrm{trop}}(J) S_i)^{\odot M_{i,J}}$$

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$$P = \sum_M \alpha_M \cdot E_M \qquad \longleftrightarrow \qquad P_{\text{trop}} = \bigoplus_M \text{trop}(\alpha_M) \odot E_{M,\text{trop}}$$

Tropical Solution

Let

$$P_{\text{trop}} = \bigoplus_{M \in \Delta} a_M \odot \epsilon_M$$

be a tropical differential polynomial. An n-tuple $S \in \mathcal{P}(\mathbb{Z}^m_{\geq 0})^n$ is called a solution of P_{trop} if for every $J \in P_{\text{trop}}(S)$ there exist $M_1, M_2 \in \Delta$ with $M_1 \neq M_2$ such that

$$J \in a_{M_1} \odot \epsilon_{M_1}(S) \quad \text{and} \quad J \in a_{M_2} \odot \epsilon_{M_2}(S).$$

Fundamental Theorem

Goal

We now want to find a relation between the solutions of the original system of differential equations and the solutions of the corresponding tropical differential polynomials.

Example 3

Let

$$P = t \cdot \frac{\partial x}{\partial t} - x.$$

The solutions of $P = 0$ are $\varphi = c\,t$, where $c \in K$. Hence, $\operatorname{Supp}(\varphi) = \{1\}$ or $\operatorname{Supp}(\varphi) = \emptyset$, respectively.

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The corresponding tropical differential polynomial is

$$P_{\mathrm{trop}}(S) = \mathrm{Vert}(\mathrm{Vert}(\{1\} + \Theta(1)S) \cup \mathrm{Vert}(S))).$$

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Let S be a solution with $0 \in S$. Then $0 \in \mathrm{Vert}(S)$ and $0 \in P_{\mathrm{trop}}(S)$. But $0 \notin \mathrm{Vert}(\{1\} + \Theta_{\mathrm{trop}}(1)S)$ in contradiction to the assumption that S is a solution.

Example 3

Since $0 \notin S$, it holds that

$$\mathrm{Vert}(\{1\} + \Theta_{\mathrm{trop}}(1)S) = \mathrm{Vert}(S)$$

and for every $J \in P_{\mathrm{trop}}(S)$ with $J \in \mathbb{Z}_{>0}$ we obtain $J \in \mathrm{Vert}(S)$ and $J \in \mathrm{Vert}(\{1\} + \Theta_{\mathrm{trop}}(1)S)$.

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This means that we do not obtain more conditions on the support of the solutions of $P = 0$ by considering P_{trop}.

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The solution $\varphi = c\,t$ of $P = t \cdot \frac{\partial x}{\partial t} - x = 0$ is also a solution of

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Since 1 cannot be an element in $(\Theta(1)P)_{\mathrm{trop}}(S)$, for a solution S it is possible that $1 \in S$. For $J \in S$ with $J \geq 2$ we obtain that J is the vertex of only one tropical differential monomial and S cannot be a solution.

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To summarize, we have obtained that

$$\mathrm{Supp}(\mathrm{Sol}(P, \Theta(1)P)) = \mathrm{Sol}(P_{\mathrm{trop}}, (\Theta(1)P)_{\mathrm{trop}}),$$

where Sol denotes the set of solutions of the implicitly defined differential equations or of the tropical differential polynomials, respectively.

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We now want to precisely state this observation as the Fundamental Theorem.

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Let K be an uncountable, algebraically closed field of characteristic zero. Let Σ be a differential ideal in the ring $K[[t_1, \ldots, t_m]]\{x_1, \ldots, x_n\}$. Then

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" $\subseteq$ " : holds for more general K.
" $\supseteq$ " : uses ultrapower construction similar to the proof of the Strong Approximation Theorem in [2].