Problems Solved:

1	2	3	4	5

Name:

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Induction-3n-2

Problem 1. Show by induction that

$$\sum_{i=1}^{n}(3i-2) = \frac{n(3n-1)}{2}$$

for $n \geq 1$.

Solution of Problem 1:

We need to show the following steps:

Base Case: $n = 1$

$$\sum_{i=1}^{1}(3i-2) = 1 = \frac{1(3 \cdot 1 - 1)}{2}$$

Done!

- Inductive Step: Fix $k \geq 1$ and assume that $\sum_{i=1}^{k}(3i-2) = \frac{k(3k-1)}{2}$ holds.
- Show that the statement holds for $k+1$:

Indeed,

$$\sum_{i=1}^{k+1}(3i-2) = \sum_{i=1}^{k}(3i-2) + 3(k+1) - 2 = \frac{k(3k-1)}{2} + 3(k+1) - 2 =$$

$$= \frac{k(3k-1) + 6(k+1) - 4}{2} = \frac{3k^2 - k + 6(k+1) - 4}{2} = \frac{3k^2 + 3k - 4k - 4 + 6(k+1)}{2} =$$

$$= \frac{3k(k+1) - 4(k+1) + 6(k+1)}{2} = \frac{3k(k+1) + 2(k+1)}{2} =$$

$$= \frac{(k+1)(3k+2)}{2} = \frac{(k+1)(3(k+1)-1)}{2}$$

Done!

Induction-m+5-n+1

Problem 2. Let $L \subseteq \mathbb{N} \times \mathbb{N}$ be the smallest set of pairs obeying the following:

- $\langle 0, 0 \rangle \in L$;
- if $\langle m, n \rangle \in L$ for some m and n, then $\langle m+5, n+1 \rangle \in L$ as well.

Show by induction that if $\langle m, n \rangle \in L$ then $m + n$ is a multiple of 3.
Hint: Recall that a natural number i is a multiple of 3 if there is some natural number j such that $i = 3 \cdot j$.

Solution of Problem 2:

We need to show the following steps:

- Base Case: $m = 0$, $n = 0$

Since $0 + 0 = 0 = 3 \cdot 0$ and $3 \cdot 0$ is a multiple of 3, we are done.

- Inductive Step: Fix m and n and assume that $\langle m, n\rangle \in L$.
- Show that the statement holds for $\langle m + 5, n + 1\rangle \in L$:

From the assumption we get that $m + n$ is a multiple of 3. This implies that there exists a natural number j such that $m + n = 3 \cdot j$.

We need to show that $m + 5 + n + 1$ is a multiple of 3.

Indeed, $m + 5 + n + 1 = m + n + 5 + 1 = 3 \cdot j + 5 + 1 = 3(j + 2)$ which implies that $m + 5 + n + 1$ is a multiple of 3.

Done!

`IndirectProof2*3` **Problem 3.** Show $\sqrt{6} \notin \mathbb{Q}$ by an indirect proof.
Hint: `http://en.wikipedia.org/wiki/Square_root_of_2#Proofs_of_irrationality`.
Hint: Note that the fact that $\sqrt{2}$ and $\sqrt{3}$ are irrational does not imply that $\sqrt{2} \cdot \sqrt{3}$ is irrational as well. For example $\sqrt{2} \cdot \sqrt{8}$ is rational although both $\sqrt{2}$ and $\sqrt{8}$ are irrational.

Solution of Problem 3:

Assume $\sqrt{6} \in \mathbb{Q}$. Then there are integers a and b with

$$\left(\frac{a}{b}\right)^2 = 6$$

and $\gcd(a, b) = 1$. Clearing denominators yields

$$a^2 = 6b^2.$$

Therefore

$$6|a^2.$$

From this, it follows that $2|a^2$ and $3|a^2$.

From this by Euklid's Lemma[1] $2|a$ and $3|a$, which implies that $6|a$. Substituting $a = 6p$ yields

$$(6p)^2 = 6b^2$$

i.e,

$$6p^2 = b^2$$

Applying the same reasoning as above, we obtain that $6|b$, and therefore $6|a$ and $6|b$. Hence, the number 6 is a common factor of a and b in contradiction to $\gcd(a, b) = 1$.

`NEA-bla-bla` **Problem 4.** Construct a nondeterministic finite state machine M over the alphabet $\{a, b, l, -\}$ such that it accepts the language $L(M) = \{bla\}$.

[1] $q|ab \implies q|a \vee q|b$ for q prime.

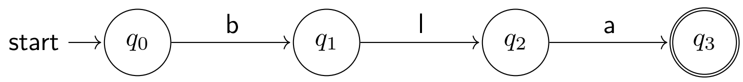
(a) Give the formal definition of M and draw the graph.

(b) What has to be changed in order for the machine to accept all finite strings of the form bla, $bla - bla$, $bla - bla - bla$ ...? (The empty word shall not be accepted.)

Solution of Problem 4:

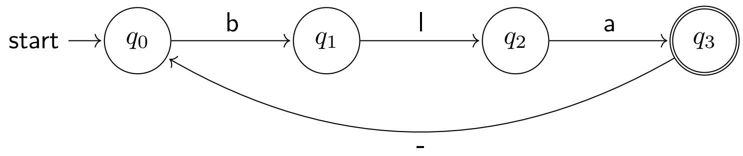
(a) $M = (\{q_0, q_1, q_2, q_3\}, \{a, b, l, -\}, \delta, q_0, \{q_3\})$, where δ is defined as follows:

$$\delta(q_0, b) = q_1, \ \delta(q_1, l) = q_2, \ \delta(q_2, a) = q_3.$$



(b) $M = (\{q_0, q_1, q_2, q_3\}, \{a, b, l, -\}, \delta, q_0, \{q_3\})$, where δ is defined as follows:

$$\delta(q_0, b) = q_1, \ \delta(q_1, l) = q_2, \ \delta(q_2, a) = q_3, \ \delta(q_3, -) = q_0.$$



NFSM010 **Problem 5.** Construct nondeterministic finite state machines for:

(a) the language L_1 of all strings over $\{0, 1\}$ that contain 010 as a substring.

(b) the language L_2 of all strings over $\{0, 1\}$ that contain the letters 0, 1, 0 in exactly that order. (Note that before, in between and after these three letters any number of other letters may occur).

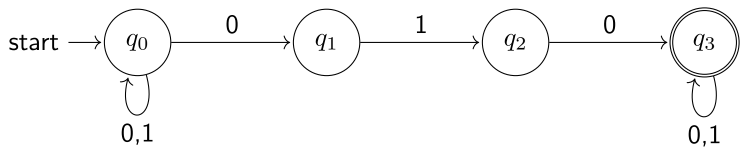
Give the formal definitions of the machines and draw their graphs.
Your two machines must not use more than 4 states. Moreover, they should only differ in their transition functions.

Solution of Problem 5:

(a) $M = (\{q_0, q_1, q_2, q_3\}, \{0, 1\}, \delta, q_0, \{q_3\})$, where δ is defined as follows:

$$\delta(q_0, 0) = \{q_0, q_1\}, \ \delta(q_0, 1) = \{q_0\}, \ \delta(q_1, 1) = \{q_2\},$$

$$\delta(q_2, 0) = \{q_3\}, \ \delta(q_3, 0) = \{q_3\}, \ \delta(q_3, 1) = \{q_3\}.$$



(b) $M = (\{q_0, q_1, q_2, q_3\}, \{0, 1\}, \delta, q_0, \{q_3\})$, where δ is defined as follows:

$$\delta(q_0, 0) = \{q_0, q_1\},\ \delta(q_0, 1) = \{q_0\},\ \delta(q_1, 0) = \{q_1\},\ \delta(q_1, 1) = \{q_1, q_2\},$$

$$\delta(q_2, 0) = \{q_2, q_3\},\ \delta(q_2, 1) = \{q_2\},\ \delta(q_3, 0) = \{q_3\},\ \delta(q_3, 1) = \{q_3\}.$$