Gruppe	Popov (8:30)	Popov (9:15)	Popov (10:15)	Hemmecke (10:15)	Hemmecke (11:00)
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Klausur 2

Berechenbarkeit und Komplexität

10. Januar 2020

Part 1 *RecFun2019*

Let $f, g : \mathbb{N} \to_P \mathbb{N}$ be two partial functions that are defined as follows:

$$f(x) = \begin{cases} \textit{undefined}, & \textit{if } x < 20, \\ 1, & \textit{if } x > 2019, \\ f(x+1), & \textit{otherwise;} \end{cases} \qquad g(x) = \begin{cases} 1, & \textit{if } x < 20, \\ \textit{undefined}, & \textit{if } x > 2019, \\ g(x+1), & \textit{otherwise.} \end{cases}$$

Let $H(x) = g(f(3x+20))$ and $h(x) = f(x) + g(x)$.

1	yes		*Is f μ-recursive?*
2	yes		*Is g Turing-computable?*
3	yes		*Is H LOOP-computable?*

Clearly, $f(x) = 1$ for all $x \geq 20$. Thus, $\forall n \in \mathbb{N} : H(n) = g(1) = 1$.

4		no	*Is there a Turing machine that halts on every input and computes h?*

$f(x)$ is undefined for $x < 20$. $g(x)$ is undefined for $x \geq 20$. Therefore, h is nowhere defined. Of course, there is a Turing maching that computes h, but if it halts it would return a value, therefore it cannot halt.

5	yes		*Does there exist a WHILE-computable function that is not primitive recursive?*

Part 2 *Grammar2019*

Consider the grammar $G = (N, \Sigma, P, S)$ where $N = \{S, A, B\}$, $\Sigma = \{0, 1\}$, $P = \{S \to 1AA0, AA \to AAA, A \to 0\}$.

6		no	*Is $100 \in L(G)$?*
7		no	*Is $L(G)$ finite?*
8		no	*Is the grammar G right-linear?*
9	yes		*Is there a deterministic finite state machine D such that $L(D) = L(G)$?*

$L(G) = L(10000^*)$ is a regular language.

10		no	*Suppose G' is a grammar, M' is a Turing machine that halts on every input word and $L(G') = L(M')$? Does then follow that G' is a context-free grammar?*

The grammar G above is a counter-example.

Part 3 *Decidable2019*

Consider the following problems. In each problem below, the input of the problem is the code $\langle M \rangle$ of a Turing machine $M = (Q, \Gamma, \sqcup, \{0, 1\}, \delta, q_0, F)$.

Problem A: *Does $L(M)$ contain a word of length greater than 2019?*

Problem B: *Does M stop on the empty word after at most 2019 steps?*

Problem C: *Is there a deterministic finite state machine D with $L(M) = L(D)$?*

Problem D: *Is there a primitive recursive function f with $L(M) = \left\{ 0^{f(n)} \mid n \in \mathbb{N} \right\}$?*

Note: For $k \in \mathbb{N}$ the expression 0^k denotes the word that consists of k letters 0.

11	yes		*Is A semi-decidable?*
12	yes		*Is B semi-decidable?*
13		no	*Is C decidable?*

Rice Theorem.

14		no	*Is D decidable?*

The property of recursively enumerable languages to be of the above form is certainly not trivial, therefore not decidable by the theorem of Rice.

15	yes		*Let f be a primitive recursive function and L be a recursive language. Let $K := \{ 0^n \mid n \in \mathbb{N} \text{ such that } 0^{f(n)} \in L \} \subseteq \{0,1\}^*$. Is the set K decidable?*

Part 4 *Complexity2019*

Let $f(n) = 20^{19 \cdot n}$, $g(n) = n^{2019}$, and $h(n) = g(\log_2(f(n))) = (19n \log_2 20)^{2019}$.

16		no	*Is it true that $f(n) = O(g(n))$?*
17	yes		*Is it true that $g(n) = \Theta(h(n))$?*
18	yes		*Is it true that $2^n = \Omega(\sqrt{2^n})$?*
19	yes		*Is it true that $(2n)^n = O(n^{2n})$?*

Part 5 *LoopWhile2019*

Let $f, g : \mathbb{N}^2 \to \mathbb{N}$ be defined as follows

$$f(a,b) := \begin{cases} 2019, & \text{if } a = b, \\ 0, & \text{otherwise;} \end{cases} \qquad g(a,b) := \begin{cases} 0, & \text{if } a = b, \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

20		no	*Are both f and g LOOP computable functions?*
21	yes		*Is (μf) a LOOP computable function?*

$(\mu f)(0) = 1$ and $(\mu f)(n) = 0$ for all $n \in \mathbb{N} \setminus \{0\}$.

22		no	*Is $(\mu g)(1) = 0$?*

Since $g(0,1)$ is undefined, $(\mu g)(1)$ is undefined, as well.

23	yes		*Is (μg) a WHILE computable functions?*

Part 6 *OpenComputability2019*

Let $L = \{ 0^k 1 0^k \mid k \in \mathbb{N} \} \subseteq \{0,1\}^*$. For a Turing machine M we denote by

- *$t_M(n)$ the maximal number of steps that M takes until it accepts or rejects a word of length n and by*
- *$s_M(n)$ the maximal distance that the head of M moves away from the beginning of the tape during the computation on an input word of length n.*

24	*1 Point*	*Determine the asymptotic time complexity of a Turing machine M with minimum time complexity that accepts L: $t_M(n) = \Theta($ $)$.*

$t_M(n) = \Theta(n^2)$

25	*1 Point*	*Determine the asymptotic space complexity of a Turing machine M with minimum space complexity that accepts L: $s_M(n) = \Theta($ $)$.*

$s_M(n) = \Theta(n)$