Distributed Memory Algorithms

Wolfgang Schreiner
Research Institute for Symbolic Computation (RISC-Linz)
Johannes Kepler University, A-4040 Linz, Austria

Wolfgang.Schreiner@risc.uni-linz.ac.at
http://www.risc.uni-linz.ac.at/people/schreine

SIMD Mesh Matrix Multiplication

Single Instruction, Multiple Data

- n^2 processors,
- $3n$ time.

Algorithm: see slide.

SIMD Mesh Matrix Multiplication

1. Precondition array
   - Shift row i by $i-1$ elements west,
   - Shift column j by $j-1$ elements north.
2. Multiply and add

   On processor $\langle i, j \rangle$:

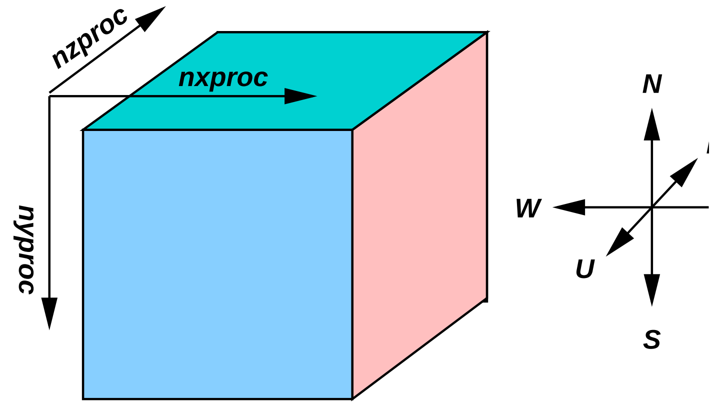
$$c = \sum_k a_{ik} * b_{kj}$$

- Inverted dimensions
  - Matrix $\downarrow i, \rightarrow j$.
  - Processor array $\downarrow$ `iyproc`, $\rightarrow$ `ixproc`.
- n shift and n arithmetic operations.
- n^2 processors.

Maspar program: see slide.

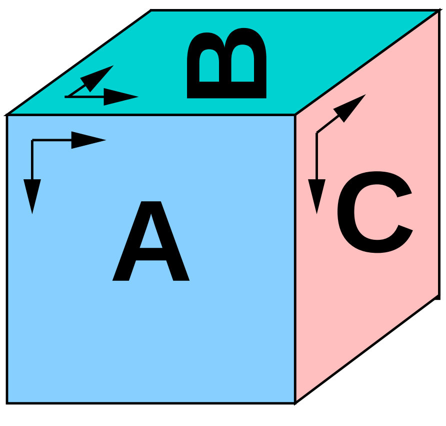
SIMD Cube Matrix Multiplication

Cube of d^3 processors



Idea

- Map $A(i, j)$ to all $P(j, i, k)$
- Map $B(i, j)$ to all $P(i, k, j)$



SIMD Cube Matrix Multiplication

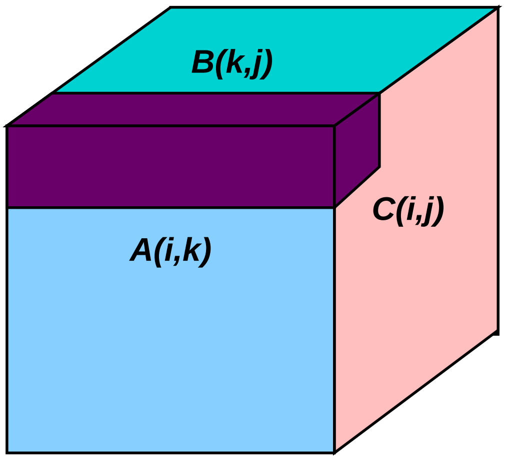
Multiplication and Addition

- Each processor computes single product

  $$P_{ijk} : c_{ijk} = a_{ik} * b_{kj}$$

- Bars along x-directions are added

  $$P_{0ij} : C_{ij} = \Sigma_k \, c_{ijk}$$



SIMD Cube Matrix Multiplication

Maspar Program

```
int A[N,N], B[N,N], C[N,N];
plural int a, b, c;

a = A[iyproc, ixproc];
b = B[ixproc, izproc];
c = a*b;

for (i = 0; i < N-1; i++)
  if (ixproc > 0)
    c = xnetE[1].c
  else
    c += xnetE[1].c;

if (ixproc == 0) C[iyproc, izproc] = c;
```

- $O(n^3)$ processors,
- $O(n)$ time.

SIMD Cube Matrix Multiplication

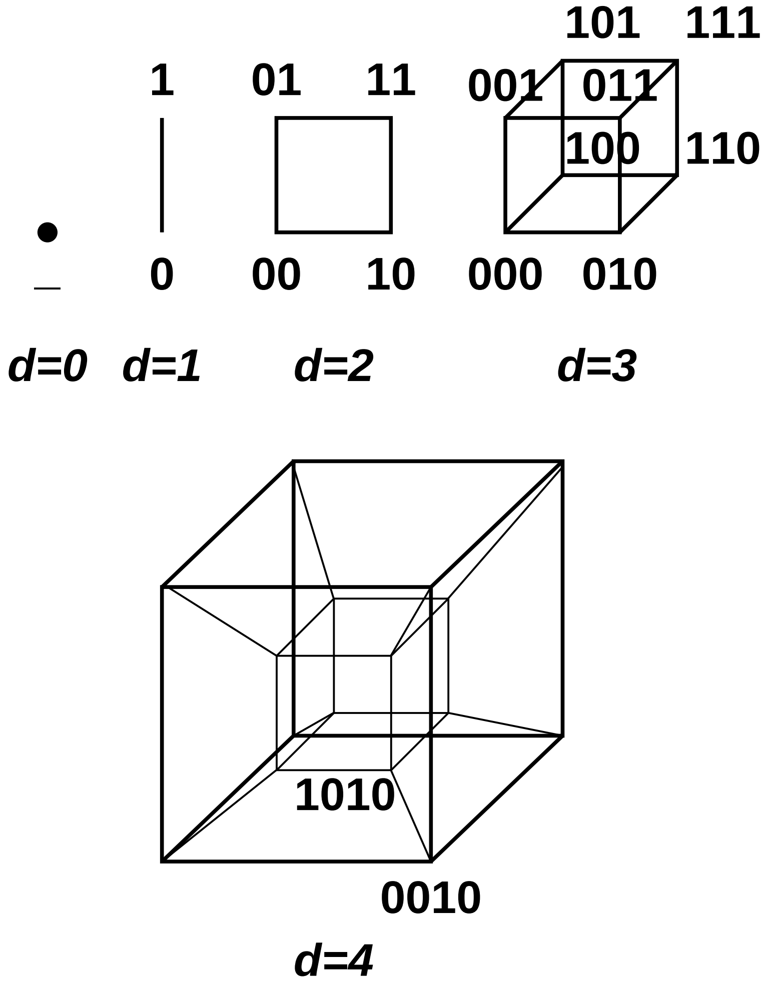
Tree-like summation

```
plural x, d;
...

x = ixproc;
d = 1;
while (d < N) {
  if (x % 2 != 0) break;
  c += xnetE[d].c;
  x /= 2;
  d *= 2;
}

if (ixproc == 0) C[iyproc, izproc] = c;
```

- $O(\log n)$ time
- $O(n^3)$ processors

Long-distance communication required!

SIMD Hypercube Mat. Multiplication



- d-dimensional hypercube $\Rightarrow$ processors indexed with d bits.
- p_1 and p_2 differ in i bits $\Rightarrow$ shortest path between p_1 and p_2 has length i.

SIMD Hypercube Matrix Multiplication

Mapping of cube with dimension n to hypercube with dimension d.

- Hypercube of $n^3 = 2^d$ processors $\Rightarrow d = 3s$ (for some s).
- 64 processors $\Rightarrow n = 4, d = 6, s = 2$.

Hypercube	$\underline{d_5 d_4}$	$\underline{d_3 d_2}$	$\underline{d_1 d_0}$
Cube	x	y	z

- Embedding algorithm
  - Cube indices in binary form (s bits each)
  - Concatenate indices ($3s = d$ bits)
- Better: use Gray code G (see later)

$\underline{d_5 d_4}$	$\underline{d_3 d_2}$	$\underline{d_1 d_0}$
$G(x)$	$G(y)$	$G(z)$

  - Neighbor processors in cube remain neighbors in hypercube.
  - Any cube algorithm can be executed with same efficiency on hypercube.

SIMD Hypercube Matrix Multiplication

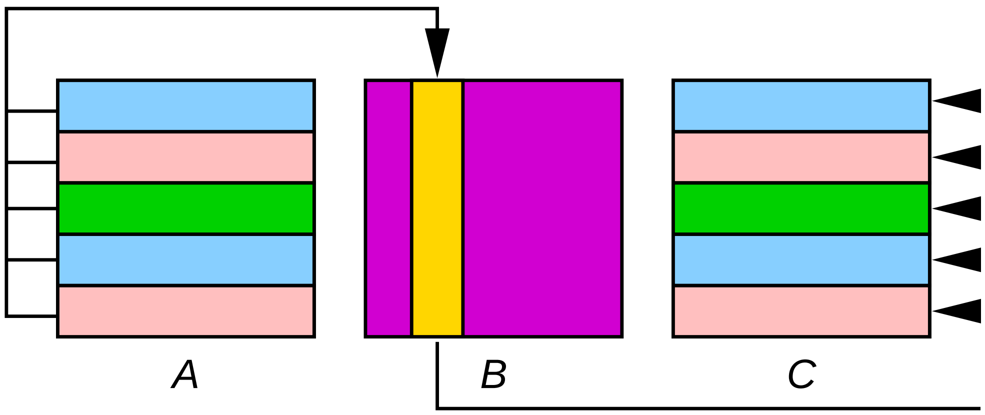
Tree summation in hypercube.

Processors	000	001	010	011	100	101	110	111
Step 1	r_0	s_0	r_1	s_1	r_2	s_2	r_3	s_3
Step 2	r_0		s_0		r_1		s_1	
Step 3	r_0				s_0			

- Each processor receives value from neighboring processors only.
- Only short-distance communication is required.

Cube algorithm can be more efficient on hypercube!

Row/Column-Oriented Matrix Multiplication



1. Load A_i on every processor P_i.
2. For all P_i do:

```
for j=0 to N-1
  Receive B_j from root
  C_ij = A_i * B_j
```

3. Collect C_i

Broadcasting of each B_j $\Rightarrow$ Step 2 takes $O(N \log N)$ time.

Ring Algorithm

See Quinn, Figure 7-15.

- Change order of multiplication by
- Using a *ring* of processors.

1. Load A_i and B_i on every processor P_i.
2. For all P_i do:

```
p = (i+1) mod N
j = i
for k=0 to N-1 do
  C_ij = A_i * B_j
  j = (j+1) mod N
  Receive B_j from P_p
```

3. Collect C_i

Point-to-point communication ⇒ Step 2 takes $O(N)$ time.

Hypercube Algorithm

Problem: How to embed ring into hypercube?

- Simple solution $H(i) = i$:
  - Ring processor i is mapped to hypercube processor $H(i)$.
  - Massive non-neighbor communication!
- How to preserve neighbor-to-neighbor communication? (see Quinn, Figure 5-13)
- Requirements for $H(i)$:
  - H must be a 1-to-1 mapping.
  - $H(i)$ and $H(i + 1)$ must differ in 1 bit.
  - $H(0)$ and $H(N - 1)$ must differ in 1 bit.

Can we construct such a function H?

Ring Successor

Assume H is given.

- Given: hypercube processor number i
- Wanted: “ring successor” $S(i)$

$$S(i) = \begin{cases} 0, & \text{if } i = N-1 \\ H(H^{-1}(i)+1), & \text{otherwise} \end{cases}$$

Same technique for embedding a 2-D (or even n-D) mesh into an hypercube (see Quinn, Figure 5-14).

Gray Codes

Recursive construction.

- 1-bit Gray code G_1

i	$G_1(i)$
0	0
1	1

- n-bit Gray code G_n

i	$G_n(i)$	i	$G_n(i)$
0	$0G_{n-1}(0)$	$n-1$	$1G_{n-1}(0)$
1	$0G_{n-1}(1)$	$n-2$	$1G_{n-1}(1)$
$\ldots$	$\ldots$	$\ldots$	$\ldots$
$\frac{n}{2}-1$	$0G_{n-1}(\frac{n}{2}-1)$	$\frac{n}{2}$	$1G_{n-1}(\frac{n}{2}-1)$

- Required properties preserved by construction!

$H(i) = G(i) = i$ *xor* $\frac{i}{2}$.

Gray Code Computation

C functions.

- Gray-Code

```
int G(int i)
{
  return(i ^ (i/2));
}
```

- Inverse Gray-Code

```
int G_inv(int i)
{
  int answer, mask;
  answer = i;
  mask = answer/2;
  while (mask > 0)
  {
    answer = answer ^ mask;
    mask = mask / 2;
  }
  return(answer);
}
```

Block-Oriented Algorithm

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

$$C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} =$$

$$\begin{pmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{pmatrix}$$

- Use block-oriented distribution introduced for shared memory multiprocessors.

  Block-matrix multiplication is analogous to scalar matrix multiplication.

- Use staggering technique introduced for 2D SIMD mesh.

  Rotation along rows and columns.

- Perform the SIMD matrix multiplication algorithm on whole *submatrices*.

  Submatrices are multiplied and shifted.

Analysis of Algorithm

n^2 matrix, p processors.

- Row/Column-oriented
  - Computation: $n^2/p * n/p = n^3/p^2$.
  - Communication: $2(\lambda + \beta n^2/p)$
  - p iterations.
- Block-oriented (staggering ignored)
  - Computation: $(n/\sqrt{p})^3 = n^3/(p\sqrt{p})$.
  - Communication: $4(\lambda + \beta n^2/p)$
  - $\sqrt{p} - 1$ iterations.
- Comparison

  $2p(\lambda + \beta n^2/p) > 4(\sqrt{p} - 1)(\lambda + \beta n^2/p)$
  $2\lambda p + 2\beta n^2 > 4\lambda(\sqrt{p} - 1) + 4\beta(\sqrt{p} - 1)n^2/p$

  1. $p > 2(\sqrt{p} - 1)$
  2. $1 > 2(\sqrt{p} - 1)/p$

  True for all $p \geq 1$.

Also including staggering, for larger p the block-oriented algorithm performs better!