Problems Solved:

21	22	23	24	25

Name:

Matrikel-Nr.:

Problem 21. Let $f : \mathbb{N} \to \mathbb{N}$ be the (partial) function

$$f(x) = \begin{cases} y & \text{such that } x = y^2 \text{ if such a } y \text{ exists,} \\ \text{undefined} & \text{if there is no } y \text{ with } x = y^2. \end{cases}$$

1. Is f loop computable? (Justify your answer.)
2. Is f a primitive recursive function? (Justify your answer.)
3. Define f by using the base functions, composition, the primitive recursion scheme, and μ-recursion (Definitions 29 and 30 from the lecture notes). Additionally you are allowed to use the (primitive recursive) functions

   $$m : \mathbb{N}^2 \to \mathbb{N}, \quad (x, y) \mapsto x \cdot y$$

   and $u : \mathbb{N}^2 \to \mathbb{N}$,

   $$u(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y. \end{cases}$$

   Other functions or rules are forbidden.
4. Why do you need the μ-recursion in your construction?
5. Is your construction in Kleene's normal form? If it is not, describe an (informal) procedure how one can turn it into Kleene's normal form.

Problem 22. Let $f : \mathbb{N} \to \mathbb{N}$ be the function

$$f(x) = \begin{cases} y & \text{such that } x = y^2 \text{ if such a } y \text{ exists,} \\ 0 & \text{if there is no } y \text{ with } x = y^2. \end{cases}$$

1. Is f loop computable? (Justify your answer.)
2. Is f a primitive recursive function? (Justify your answer.)
3. Define f by using the base functions, composition and the primitive recursion scheme. Additionally you are allowed to use the (primitive recursive) functions

   $$m : \mathbb{N}^2 \to \mathbb{N}, \quad (x, y) \mapsto x \cdot y$$

   $u : \mathbb{N}^2 \to \mathbb{N}$,

   $$u(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y. \end{cases}$$

   and $IF : \mathbb{N}^3 \to \mathbb{N}$,

   $$IF(x, y, z) = \begin{cases} y & \text{if } x = 0 \\ z & \text{otherwise.} \end{cases}$$

Problem 23. Prove *formally* by only employing Definition 29, that if $t, e, c : \mathbb{N} \to \mathbb{N}$ are primitive recursive, then also $f : \mathbb{N} \to \mathbb{N}$ is primitive recursive where f is defined by

$$f(x) = \begin{cases} t(x) & \text{if } c(x) = 0 \\ e(x) & \text{otherwise.} \end{cases}$$

Do this, by explicitly defining f in terms of composition and the primitive recursion scheme from the given function and some helper functions (which you must prove to be primitive recursive, as well).
Hint: Show that the function $IF : \mathbb{N}^3 \to \mathbb{N}$,

$$IF(x, y, z) = \begin{cases} y & \text{if } x = 0 \\ z & \text{otherwise.} \end{cases}$$

is primitive recursive. Then use it to show that f is primitive recursive.

Problem 24. Let P be the following program for counting how many of the first n odd numbers starting with 3 are prime.

```
s :=0
i :=3
LOOP n DO
    p = isprime(i)
    IF p = 1 THEN s := s+1
      END;
    i := i+2
END;
```

Convert P into primitive recursive function, provided *isprime* is a given loop program (you may assume that a corresponding primitive recursive function *isprime* is given as well).

Problem 25. For the function

$$f(n) = \begin{cases} 3n + 1 & \text{if } n \text{ is odd,} \\ \frac{n}{2} & \text{if } n \text{ is even.} \end{cases}$$

let ν be defined as

$$\nu(n) = (\mu p)(n)$$

where p, and h are given through

$$\begin{aligned} p(k, n) &= h(k, n) - 1, \\ h(k, n) &= \begin{cases} n & \text{if } k = 0 \\ f(h(k-1, n)) & \text{if } k > 0. \end{cases} \end{aligned}$$

The function ν determines the minimal number of applications of f that must be done to end up in the value 1. In other words, $\nu(n)$ is the minimal natural number k for which

$$f^k(n) = \underbrace{f(f(\cdots f(}_{k\text{-fold}} n) \cdots) = 1.$$

1. Compute $\nu(3)$.
2. The above definitions of p and h are not formally defined as primitive functions according to Definition 29 of the lecture notes. Fill in the details.
3. Construct a while program that computes ν. You may use any abbreviation for while or loop programs from the lecture notes as long as you explicitly refer to it via page number.
4. Based on what you read (e.g. in Wikipedia) about the Collatz-Problem (or Collatz conjecture), do you think that p is primitive recursive or not? Justify your answer.