On quantitative monadic first-order logic

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Joint work with Eleni Mandrali

Outline

- A fundamental result on language theory relating *LTL*-definability, *FO*-logic-definability, star-freeness, counter-freeness

- Over idempotent zero-divisor free totally commutative complete semirings:

  - *LTL*

  - *FO* logic

  - (ω-)star-free series

  - Counter-free (Büchi) automata

  - The main result

- Open problems - Future research

Words

- *alphabet* A : is a finite set
- $A^* = \{\varepsilon\} \cup \{a_0 \dots a_{n-1} \mid a_0, \dots, a_{n-1} \in A\}$ *finite words* over A
- $w = a_0 \dots a_{n-1}, \;\; dom(w) = \{0, 1, \dots, n-1\}$
- $w = w(0) \dots w(n-1)$
- $A^\omega = \{a_0 a_1 \dots \mid a_0, a_1, \dots \in A\}$ *infinite words* over A
- $w = a_0 a_1 \dots, \qquad dom(w) = \omega (= \mathbb{N})$
- $w = w(0)w(1)\dots$
- $w_{\geq i} = w(i)w(i+1)\dots, \;\; (i \geq 0)$

Words - languages

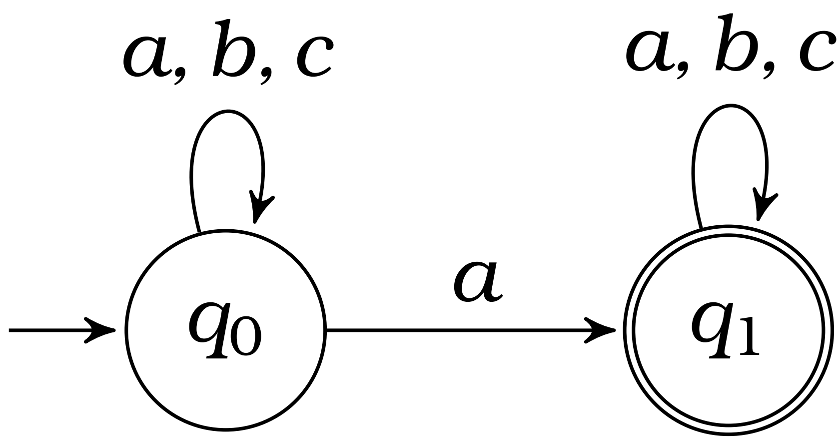
$L \subseteq A^*$ (finitary) language

$L \subseteq A^\omega$ infinitary language

Example

- $A = \{a, b, c\}$
- L the (finitary) language of words with at least one occurrence of a

Automaton



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- *Expression:* $A^* a A^*$
- *Monadic first-order logic sentence:* $\exists x . P_a(x)$
- *Linear temporal logic (LTL) formula:* $\Diamond p_a$

Büchi automata

- A (nondeterministic) *Büchi automaton*

$$\mathcal{A} = (Q, A, I, \Delta, F)$$

  - Q: the *finite state set*
  - A: the *input alphabet*
  - $I \subseteq Q$: the *initial state set*
  - $\Delta \subseteq Q \times A \times Q$: the *set of transitions*
  - $F \subseteq Q$: the *final state set*

- $w = a_0 a_1 \ldots \in A^\omega$
- *path of $\mathcal{A}$ over w*

$$P_w = (q_0, a_0, q_1)(q_1, a_1, q_2) \ldots \in \Delta^\omega$$

- P_w: *successful* if $q \in F$ occurs infinitely often

Counter-free automata

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- $q \in Q$, $w \in A^*$, $n \geq 1$, if there is a path

$$q \xrightarrow{w^n} q$$

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- then there is a path

$$q \xrightarrow{w} q$$

FO logic - Syntax

- **Syntax**

$$\varphi ::= true \mid P_a(x) \mid x \leq y \mid \neg\varphi \mid \varphi \vee \varphi \mid \exists x \centerdot \varphi$$

$a \in A$, x, y first-order variables

- $false = \neg true$
- $\neg\neg\varphi = \varphi$
- $\varphi \wedge \psi = \neg(\neg\varphi \vee \neg\psi)$
- $\forall x \centerdot \varphi = \neg(\exists x \centerdot \neg\varphi)$

FO logic - Semantics

- φ FO logic formula, $w \in A^\omega$
- first-order variables in φ represent *positions* in w
- $(w, \mathit{free}(\varphi))$*-assignment:* $\sigma : \mathit{free}(\varphi) \to \omega (= \mathit{dom}(w))$
- $i \in \omega$
- $\sigma[x \to i] : \mathit{free}(\varphi) \cup \{x\} \to \omega$
  coincides with σ on $\mathit{free}(\varphi) \setminus \{x\}$

FO logic - Semantics

- $(w, \sigma) \models \varphi$ by induction on the structure of φ:
  - $(w, \sigma) \models true$
  - $(w, \sigma) \models P_a(x)$ iff $w(\sigma(x)) = a$
  - $(w, \sigma) \models x \leq y$ iff $\sigma(x) \leq \sigma(y)$
  - $(w, \sigma) \models \neg\varphi$ iff $(w, \sigma) \not\models \varphi$
  - $(w, \sigma) \models \varphi \vee \psi$ iff $(w, \sigma) \models \varphi$ or $(w, \sigma) \models \psi$
  - $(w, \sigma) \models \exists x \,.\, \varphi$ iff there exists $i \in \omega$ such that $(w, \sigma[x \rightarrow i]) \models \varphi$

Linear Temporal Logic (LTL) - Syntax

For every $a \in A$ we consider an atomic proposition p_a

$AP = \{p_a \mid a \in A\}$

Syntax

$$\varphi ::= true \mid p_a \mid \neg\varphi \mid \varphi \vee \varphi \mid \bigcirc\varphi \mid \varphi U \varphi$$

Linear Temporal Logic (LTL) - Semantics

- **Semantics** $w \in A^{\omega}$
  - $w \models p_a$ iff $w(0) = a$
  - $w \models \neg\varphi$ iff $w \not\models \varphi$
  - $w \models \varphi \vee \psi$ iff $w \models \varphi$ or $w \models \psi$
  - $w \models \bigcirc\varphi$ iff $w_{\geq 1} \models \varphi$
  - $w \models \varphi U \psi$ iff there exists $i \geq 0$ such that ($w_{\geq i} \models \psi$ and $w_{\geq j} \models \varphi$ for every $0 \leq j < i$)
- Further formulas $\Diamond\varphi := true U \varphi$, $\Box\varphi := \neg\Diamond\neg\varphi$
- **Example:** $A = \{a, b, c\}$, $w = b^3 a^2 c^{\omega} \models p_b U p_a$ $(i = 3)$

$w' = c^{30} b^{\omega} \not\models \Diamond p_a$

Star-free languages

- The class of star-free languages over A is the smallest family of languages over A which contains $\emptyset$, the singleton $\{a\}$ for every $a \in A$, and it is closed under finite union, complement and concatenation.

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  - A^ω, $(ab)^*A^\omega$
- $(aa)^*$ is **not** star-free

A fundamental result

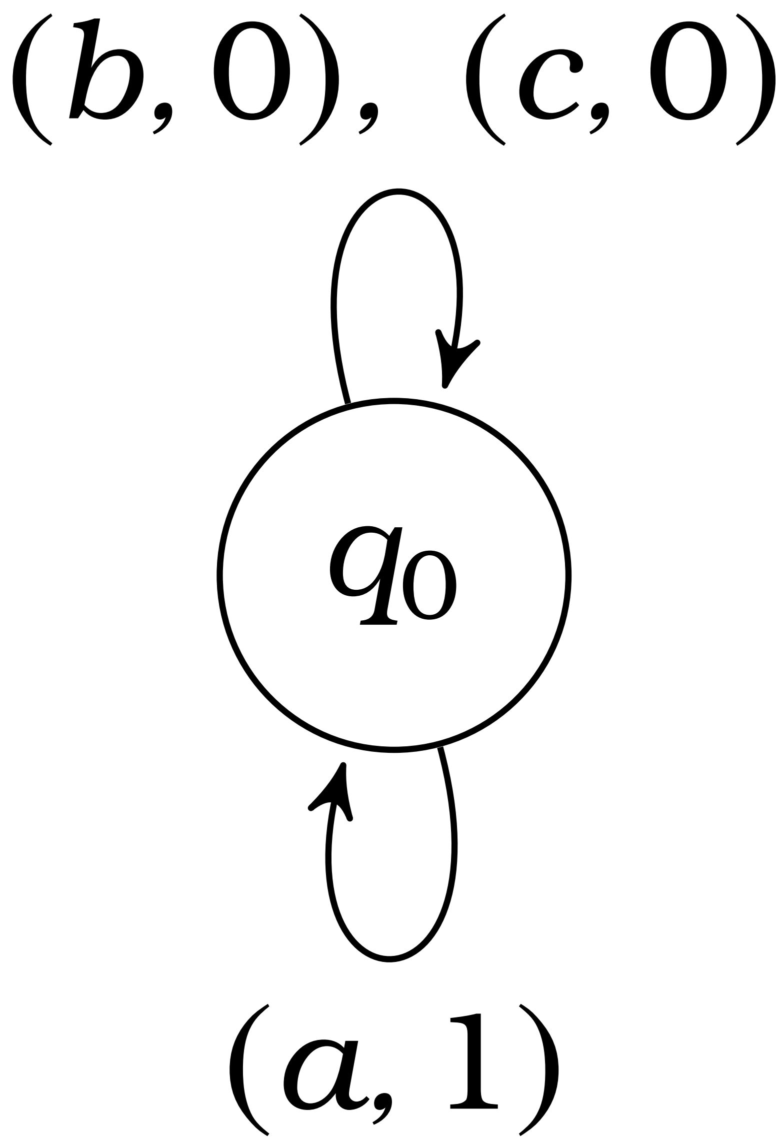
- A alphabet, $L \subseteq A^*$ (resp. $L \subseteq A^\omega$)
- The following are equivalent:
  - L is definable in FO logic
  - L is definable in LTL
  - L is star-free (resp. ω-star-free)
  - L is accepted by a counter-free (resp. counter-free Büchi) automaton
- V. Diekert and P. Gastin, First-order definable languages (2007)

Weights

Example

- $A = \{a, b, c\}$
- L the (finitary) language of words with at least one occurrence of a
- What about the *number of occurrences* of a in a word?
- $w = bcabaca$ Number of a's: 3

Weighted automaton



Semirings

- $(K, +, \cdot, 0, 1)$: semiring (simply denoted by K)
  - $+$ binary associative and commutative operation on K, neutral element 0
    - $k + (l + m) = (k + l) + m$
    - $k + l = l + k$
    - $k + 0 = k$
  - $\cdot$ binary associative operation on K, neutral element 1
    - $k \cdot (l \cdot m) = (k \cdot l) \cdot m$
    - $k \cdot 1 = 1 \cdot k = 1$
  - $\cdot$ distributes over $+$
    - $k \cdot (l + m) = k \cdot l + k \cdot m$
    - $(k + l) \cdot m = k \cdot m + l \cdot m$
  - $k \cdot 0 = 0 \cdot k = 0$
- K commutative if $\cdot$ is commutative

Semirings

- semiring K
- *idempotent*

$$k + k = k$$

zero-divisor free

$$k \cdot k' = 0 \implies k = 0 \text{ or } k' = 0$$

$k, k' \in K$.

Complete semirings

- K *complete* $\sum_I : K^I \to K$ (I index set):
  - $\sum_{i\in\emptyset} k_i = 0$
  - $\sum_{i\in\{j\}} k_i = k_j$
  - $\sum_{i\in\{j,l\}} k_i = k_j + k_l \quad j \neq l$
  - $\sum_{j\in J} \sum_{i\in I_j} k_i = \sum_{i\in I} k_i \quad \bigcup_{j\in J} I_j = I, \;\; I_j \cap I_{j'} = \emptyset$
  - $\sum_{i\in I}(k \cdot k_i) = k \cdot (\sum_{i\in I} k_i)$
  - $\sum_{i\in I}(k_i \cdot k) = (\sum_{i\in I} k_i) \cdot k$

Totally complete semirings

- K *totally complete* $\prod_{i \geq 0} : K^i \to K$:
  - $\prod_{i \geq 0} 1 = 1$
  - $\prod_{i \geq 0} k_i = \prod_{i \geq 0} k_i'$

    $k_0' = k_0 \cdot \ldots \cdot k_{n_1}, k_1' = k_{n_1+1} \cdot \ldots \cdot k_{n_2}, \ldots$
  - $k_0 \cdot \prod_{i \geq 0} k_{i+1} = \prod_{i \geq 0} k_i$
  - $\prod_{j \geq 1} \sum_{i \in I_j} k_i = \sum_{(i_1, i_2, \ldots) \in I_1 \times I_2 \times \ldots} \prod_{j \geq 1} k_{i_j}$
  - $k \neq 0 \implies \prod_{i \geq 0} k \neq 0$
- K *totally commutative complete*

$$\prod_{i \geq 0} (k_i \cdot k_i') = \left(\prod_{i \geq 0} k_i \right) \cdot \left(\prod_{i \geq 0} k_i' \right).$$

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  - $\prod_{i\geq 0} 1 = 1$
  - $\prod_{i\geq 0} k_i = \prod_{i\geq 0} k'_i$

    $k'_0 = k_0 \cdot \ldots \cdot k_{n_1}, k'_1 = k_{n_1+1} \cdot \ldots \cdot k_{n_2}, \ldots$
  - $k_0 \cdot \prod_{i\geq 0} k_{i+1} = \prod_{i\geq 0} k_i$
  - $\prod_{j\geq 1} \sum_{i\in I_j} k_i = \sum_{(i_1,i_2,\ldots)\in I_1\times I_2\times\ldots} \prod_{j\geq 1} k_{i_j}$
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Idempotent, zero-divisor free totally commutative complete semirings

Examples

- the *arctical semiring* or *max-plus semiring* with $+\infty$
  $(\mathbb{R}_+ \cup \{\pm\infty\}, \max, +, -\infty, 0)$
- each complete chain, in particular the *fuzzy semiring*
  $F = ([0,1], \sup, \inf, 0, 1)$

Weighted LTL over K - Syntax

- Atomic propositions: $AP = \{p_a \mid a \in A\}$.

Definition

Syntax of *weighted LTL formulas*

$$\varphi ::= k \mid p_a \mid \neg\varphi \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \bigcirc\varphi \mid \varphi U \varphi \mid \Box\varphi$$

$k \in K$, $p_a \in AP$.

- $LTL(K, A)$: weighted LTL formulas
- $\varphi \in LTL(K, A)$: *boolean* if it has no weights $\neq 0, 1$

Weighted LTL over A and K - Semantics

Definition

$\varphi \in LTL(K, A)$, **semantics** $\|\varphi\| : A^\omega \to K$, $(w \in A^\omega)$

- $(\|k\|, w) = k$,
- $(\|p_a\|, w) = \begin{cases} 1 & \text{if } w(0) = a \\ 0 & \text{otherwise} \end{cases}$,
- $(\|\neg\varphi\|, w) = \begin{cases} 1 & \text{if } (\|\varphi\|, w) = 0 \\ 0 & \text{otherwise} \end{cases}$,
- $(\|\varphi \vee \psi\|, w) = (\|\varphi\|, w) + (\|\psi\|, w)$,
- $(\|\varphi \wedge \psi\|, w) = (\|\varphi\|, w) \cdot (\|\psi\|, w)$,
- $(\|\bigcirc\varphi\|, w) = (\|\varphi\|, w_{\geq 1})$,
- $(\|\varphi U \psi\|, w) = \sum_{i \geq 0} \left(\left(\prod_{0 \leq j < i} (\|\varphi\|, w_{\geq j}) \right) \cdot (\|\psi\|, w_{\geq i}) \right)$,
- $(\|\Box\varphi\|, w) = \prod_{i \geq 0} (\|\varphi\|, w_{\geq i})$.

Weighted LTL over A and K - Some formulas

- *almost boolean LTL formula*: $\varphi = \bigwedge_{1 \leq i \leq n} \varphi_i$

  φ_i is boolean or $\varphi_i = \bigvee_{a \in A} (k_a \wedge p_a)$

- $abLTL(K, A)$: almost boolean LTL formulas

Weighted *LTL* over *A* and *K* - *ULTL*-fragment

Definition

$ULTL(K,A)$ *U-nesting LTL formulas:*

- $k \in ULTL(K,A)$ for every $k \in K$.
- $abLTL(K,A) \subseteq ULTL(K,A)$.
- If $\varphi \in ULTL(K,A)$, then $\neg\varphi \in ULTL(K,A)$.
- If $\varphi, \psi \in ULTL(K,A)$, then $\varphi \wedge \psi, \varphi \vee \psi \in ULTL(K,A)$.
- If $\varphi \in ULTL(K,A)$, then $\bigcirc\varphi \in ULTL(K,A)$.
- If φ is boolean or $\varphi = \bigvee_{a\in A}(k_a \wedge p_a)$, then $\Box\varphi \in ULTL(K,A)$.
- If $\varphi \in abLTL(K,A)$ and $\psi \in ULTL(K,A)$, then $\varphi U \psi \in ULTL(K,A)$.

Weighted *LTL* over *A* and *K* - *ULTL*-fragment

- $r : A^{\omega} \to K$
- ω-*ULTL-definable* if $r = \|\varphi\|$, $\varphi \in ULTL(K, A)$
- $\omega\text{-}ULTL(K, A)$: ω-*ULTL*-definable series

Weighted FO logic over A and K - Syntax

Definition

Syntax of *weighted FO logic formulas*

$$\varphi ::= k \mid P_a(x) \mid x \leq y \mid \neg\varphi \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \exists x \,.\, \varphi \mid \forall x \,.\, \varphi$$

$k \in K$, $a \in A$.

- $FO(K, A)$: weighted FO logic formulas
- $\varphi \in FO(K, A)$: *boolean* if it has no weights $\neq 0, 1$

Weighted *FO* logic over *A* and *K* - Semantics

Definition

$\varphi \in FO(K,A)$, **semantics** $\|\varphi\| : \left(A \times \{0,1\}^{free(\varphi)}\right)^{\omega} \to K$

- $(\|k\|, (w,\sigma)) = k$,
- $(\|P_a(x)\|, (w,\sigma)) = \begin{cases} 1 & \text{if } w(\sigma(x)) = a \\ 0 & \text{otherwise} \end{cases}$,
- $(\|x \leq y\|, (w,\sigma)) = \begin{cases} 1 & \text{if } \sigma(x) \leq \sigma(y) \\ 0 & \text{otherwise} \end{cases}$,
- $(\|\neg\varphi\|, (w,\sigma)) = \begin{cases} 1 & \text{if } (\|\varphi\|, (w,\sigma)) = 0 \\ 0 & \text{otherwise} \end{cases}$,
- $(\|\varphi \vee \psi\|, (w,\sigma)) = (\|\varphi\|, (w,\sigma)) + (\|\psi\|, (w,\sigma))$,
- $(\|\varphi \wedge \psi\|, (w,\sigma)) = (\|\varphi\|, (w,\sigma)) \cdot (\|\psi\|, (w,\sigma))$,

Weighted *FO* logic over *A* and *K* - Semantics

Definition (continued)

- $(\|\exists x \,.\, \varphi\|, (w,\sigma)) = \sum_{i \geq 0} (\|\varphi\|, (w, \sigma[x \to i]))$,
- $(\|\forall x \,.\, \varphi\|, (w,\sigma)) = \prod_{i \geq 0} (\|\varphi\|, (w, \sigma[x \to i]))$.

Weighted FO logic over A and K- $WQFO$-fragment

Definition

$\varphi \in FO(K, A)$ *weakly quantified* if whenever φ

contains a subformula of the form $\forall x \, . \, \psi$, then

- ψ is boolean formula, or
- $\psi = \bigvee_{a \in A} (k_a \wedge P_a(x))$, $\quad k_a \in K$ or
- $\psi = y \leq x \rightarrow \bigvee_{a \in A} (k_a \wedge P_a(x))$, $\quad k_a \in K$ or
- $\psi = z \leq x < y \rightarrow \bigvee_{a \in A} (k_a \wedge P_a(x))$, $\quad k_a \in K$

- $s : A^\omega \rightarrow K$ $\;$ *ω-wqFO-definable* if $s = \|\varphi\|$
  φ $\quad$ weakly quantified sentence
- ω-$wqFO(K, A)$: $\;$ *ω-wqFO*-definable series

Star-free series

- *monomials:* $k_a a, \quad a \in A, \quad k_a \in K$
- *letter-step series:* $s = \sum_{a \in A} k_a a,$
- *complement $\overline{s}$ of a series s:* $(\overline{s}, w) = \begin{cases} 1 & \text{if } (s, w) = 0 \\ 0 & \text{otherwise} \end{cases}$
- *Hadamard product of series r and s :*

$$(r \odot s, w) = (r, w) \cdot (s, w)$$

- *Cauchy product of series r and s :*

$$(r \cdot s, w) = \sum_{u,v \in A^*, w = uv} (r, u) \cdot (s, v)$$

Star-free series

- The nth-iteration r^n $(n \geq 0)$ of $r : A^* \to K$
- $r^0 = 1_\varepsilon$ and $r^{n+1} = r \cdot r^n$ for $n \geq 0$
- $(r^n, w) = \sum_{u_i \in A^*, w = u_1 \ldots u_n} \left(\prod_{1 \leq i \leq n} (r, u_i) \right)$
- r *proper* if $(r, \varepsilon) = 0$
- *iteration* r^+ *of a proper series* r : $r^+ = \sum_{n>0} r^n$

Star-free series

- $r : A^* \to K \quad s : A^\omega \to K$
- *Cauchy product of r and s :*
- $$(r \cdot s, w) = \sum_{u \in A^*, v \in A^\omega, w = uv} (r, u) \cdot (s, v)$$
- *ω-iteration of a proper finitary series r :* $\quad r^\omega : A^\omega \to K$
- $$(r^\omega, w) = \sum_{u_i \in A^*, w = u_1 u_2 \ldots} \left(\prod_{i \geq 1} (r, u_i) \right)$$

Star-free series and ω-star-free series

Definition

The class of *star-free series over* A *and* K, denoted by $SF(K, A)$, is the least class of series containing the monomials (over A and K) and being closed under sum, Hadamard product, complement, Cauchy product, and iteration restricted to letter-step series.

Definition

The class of ω*-star-free series over* A *and* K, denoted by ω-$SF(K, A)$, is the least class of infinitary series generated by the monomials (over A and K) by applying finitely many times the operations of sum, Hadamard product, complement, Cauchy product, iteration restricted to letter-step series, and ω-iteration restricted to letter-step series.

Weighted automata

- A weighted automaton over A and K : $\mathcal{A} = (Q, in, wt, F)$ where
  - Q is the *finite state set*,
  - $in : Q \to K$ is the *initial distribution*,
  - $wt : Q \times A \times Q \to K$ assigns *weights* to the transitions,
  - $F \subseteq Q$ is the *final state set*.
- $w = a_0 \ldots a_{n-1} \in A^*$, a path: $P_w := ((q_i, a_i, q_{i+1}))_{0 \leq i \leq n-1}$
- *running weight of* P_w

$$rwt(P_w) := \prod_{0 \leq i \leq n-1} wt((q_i, a_i, q_{i+1}))$$

weight of P_w

$$weight(P_w) := in(q_0) \cdot rwt(P_w)$$

Weighted automata

- P_w : *successful* if $q_n \in F$
- *behavior of* $\mathcal{A}$:

$$\|\mathcal{A}\| : A^* \to K$$

$$(\|\mathcal{A}\|, w) = \sum_{P_w \text{ succ}} weight(P_w).$$

Weighted Büchi automata

- A weighted Büchi automaton $\mathcal{A} = (Q, in, wt, F)$
- $w = a_0 a_1 \ldots \in A^\omega$, a path: $P_w := ((q_i, a_i, q_{i+1}))_{i \geq 0}$
- *running weight of* P_w

$$rwt(P_w) := \prod_{i \geq 0} wt\left((q_i, a_i, q_{i+1})\right)$$

weight of P_w

$$weight(P_w) := in(q_0) \cdot rwt(P_w)$$

- P_w : *successful* if $q \in F$ occurs infinitely often along P_w
- *behavior of* $\mathcal{A}$: $\|\mathcal{A}\| : A^\omega \to K$

$$(\|\mathcal{A}\|, w) = \sum_{P_w \text{ succ}} weight(P_w).$$

Counter-free weighted automata

- $P_{(q,w,q)}$ a path of $\mathcal{A}$ from q to q over w

Definition

A weighted (resp. weighted Büchi) automaton $\mathcal{A} = (Q, in, wt, F)$ is called *counter-free* if for every $q \in Q$, $w \in A^*$, and $n \geq 1$, the relation

$$\sum_{P_{(q,w^n,q)}} rwt\left(P_{(q,w^n,q)}\right) \neq 0$$

implies

$$\sum_{P_{(q,w^n,q)}} rwt\left(P_{(q,w^n,q)}\right) = \left(\sum_{P_{(q,w,q)}} rwt\left(P_{(q,w,q)}\right)\right)^n.$$

Simple counter-free weighted automata

Definition

A counter-free weighted (resp. counter-free weighted Büchi) automaton $\mathcal{A} = (Q, in, wt, F)$ over A and K is *simple* if for every $q, q', p, p' \in Q$, and $a \in A$,

$$in(q) \neq 0 \neq in(q')$$

implies

$$in(q) = in(q'),$$

and

$$wt((q, a, q')) \neq 0 \neq wt((p, a, p'))$$

implies

$$wt((q, a, q')) = wt((p, a, p')).$$

Simple counter-free series

Definition

A series $r : A^\omega \to K$ is called *almost simple counter-free* if

$$r = \sum_{1 \leq i \leq n} \left(r_1^{(i)} \cdot \ldots \cdot r_{m_i}^{(i)} \right)$$

where, for every $1 \leq i \leq n$, $r_1^{(i)}, \ldots, r_{m_i - 1}^{(i)}$ are accepted by simple counter-free weighted automata and $r_{m_i}^{(i)}$ is accepted by a simple counter-free weighted Büchi automaton.

- $\omega\text{-}asCF(K, A)$: almost simple counter-free series

Main result

Theorem

$$\omega\text{-}ULTL(K,A) = \omega\text{-}wqFO(K,A) = \omega\text{-}SF(K,A) = \omega\text{-}asCF(K,A).$$

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Proof.

We prove the inclusions:

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$$\subseteq \omega\text{-}asCF(K,A) \subseteq \omega\text{-}ULTL\,(K,A)\,.$$

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- Thank you!