

# Specifying Properties of Concurrent Systems

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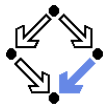


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## 1. The Basics of Temporal Logic

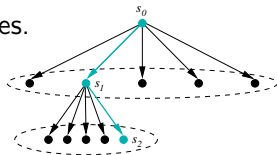
## 2. Specifying with Linear Time Logic

# Motivation



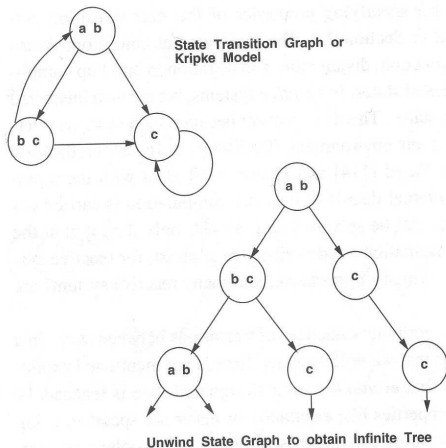
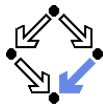
We need a language for specifying system properties.

- A system  $S$  is a pair  $\langle I, R \rangle$ .
  - Initial states  $I$ , transition relation  $R$ .
  - More intuitive: reachability graph.
    - Starting from an initial state  $s_0$ , the system runs evolve.
- Consider the reachability graph as an infinite **computation tree**.
  - Different tree nodes may denote occurrences of the same state.
    - Each occurrence of a state has a unique predecessor in the tree.
  - Every path in this tree is infinite.
    - Every finite run  $s_0 \rightarrow \dots \rightarrow s_n$  is extended to an infinite run  $s_0 \rightarrow \dots \rightarrow s_n \rightarrow s_n \rightarrow s_n \rightarrow \dots$
- Or simply consider the graph as a **set of system runs**.
  - Same state may occur multiple times (in one or in different runs).



**Temporal logic describes such trees respectively sets of system runs.**

# Computation Trees versus System Runs



Set of system runs:

$[a, b] \rightarrow c \rightarrow c \rightarrow \dots$

$[a, b] \rightarrow [b, c] \rightarrow c \rightarrow \dots$

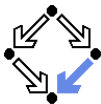
$[a, b] \rightarrow [b, c] \rightarrow [a, b] \rightarrow \dots$

$[a, b] \rightarrow [b, c] \rightarrow [a, b] \rightarrow \dots$

...

**Figure 3.1**  
Computation trees.

Edmund Clarke et al: "Model Checking", 1999.



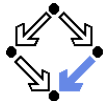
# State Formula

Temporal logic is based on classical logic.

- A **state formula**  $F$  is evaluated on a state  $s$ .
  - Any predicate logic formula is a state formula:  
 $p(x), \neg F, F_0 \wedge F_1, F_0 \vee F_1, F_0 \Rightarrow F_1, F_0 \Leftrightarrow F_1, \forall x : F, \exists x : F$ .
  - In **propositional temporal logic** only propositional logic formulas are state formulas (no quantification):  
 $p, \neg F, F_0 \wedge F_1, F_0 \vee F_1, F_0 \Rightarrow F_1, F_0 \Leftrightarrow F_1$ .
- **Semantics**:  $s \models F$  (“ $F$  holds in state  $s$ ”).
  - Example: semantics of conjunction.
    - $(s \models F_0 \wedge F_1) :\Leftrightarrow (s \models F_0) \wedge (s \models F_1)$ .
    - “ $F_0 \wedge F_1$  holds in  $s$  if and only if  $F_0$  holds in  $s$  and  $F_1$  holds in  $s$ ”.

**Classical logic reasoning on individual states.**

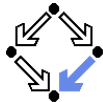
# Temporal Logic



Extension of classical logic to reason about multiple states.

- Temporal logic is an instance of **modal logic**.
  - Logic of “multiple worlds (situations)” that are in some way related.
  - Relationship may e.g. be a **temporal** one.
  - Amir Pnueli, 1977: temporal logic is suited to system specifications.
  - Many variants, two fundamental classes.
- **Branching Time Logic**
  - Semantics defined over **computation trees**.  
At each moment, there are multiple possible futures.
  - Prominent variant: **CTL**.  
Computation tree logic; a propositional branching time logic.
- **Linear Time Logic**
  - Semantics defined over **sets of system runs**.  
At each moment, there is only one possible future.
  - Prominent variant: **PLTL**.  
A propositional linear time logic.

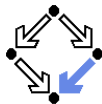
# Branching Time Logic (CTL)



We use temporal logic to specify a system property  $F$ .

- **Core question:**  $S \models F$  (" $F$  holds in system  $S$ ").
  - System  $S = \langle I, R \rangle$ , temporal logic formula  $F$ .
- **Branching time logic:**
  - $S \models F \Leftrightarrow S, s_0 \models F$ , for every initial state  $s_0$  of  $S$ .
  - Property  $F$  must be evaluated on every pair of system  $S$  and initial state  $s_0$ .
  - Given a computation tree with root  $s_0$ ,  $F$  is evaluated on **that tree**.

**CTL formulas are evaluated on computation trees.**



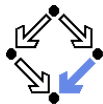
# State Formulas

We have additional state formulas.

- A **state formula**  $F$  is evaluated on state  $s$  of System  $S$ .
  - Every (classical) state formula  $f$  is such a state formula.
  - Let  $P$  denote a **path formula** (later).
    - Evaluated on a **path** (state sequence)  $p = p_0 \rightarrow p_1 \rightarrow p_2 \rightarrow \dots$   
 $R(p_i, p_{i+1})$  for every  $i$ ;  $p_0$  need not be an initial state.
  - Then the following are **state formulas**:
    - A**  $P$  (“in every path  $P$ ”),
    - E**  $P$  (“in some path  $P$ ”).
  - **Path quantifiers: A, E.**
- **Semantics:**  $S, s \models F$  (“ $F$  holds in state  $s$  of system  $S$ ”).
  - $S, s \models f \Leftrightarrow s \models f$ .
  - $S, s \models \mathbf{A} P \Leftrightarrow S, p \models P$ , for every path  $p$  of  $S$  with  $p_0 = s$ .
  - $S, s \models \mathbf{E} P \Leftrightarrow S, p \models P$ , for some path  $p$  of  $S$  with  $p_0 = s$ .



# Path Formulas



We have a class of formulas that are not evaluated over individual states.

- A **path formula**  $P$  is evaluated on a path  $p$  of system  $S$ .

- Let  $F$  and  $G$  denote **state formulas**.

- Then the following are **path formulas**:

**X**  $F$  ("next time  $F$ "),

**G**  $F$  ("always  $F$ "),

**F**  $F$  ("eventually  $F$ "),

$F$  **U**  $G$  (" $F$  until  $G$ ").

- **Temporal operators: X, G, F, U.**

- **Semantics:**  $S, p \models P$  (" $P$  holds in path  $p$  of system  $S$ ").

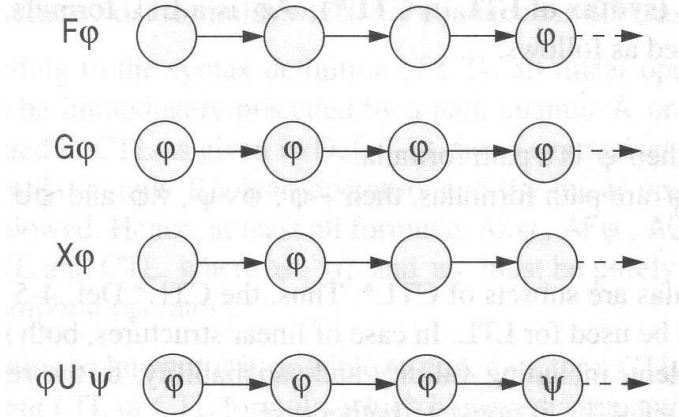
$$S, p \models \mathbf{X} F \Leftrightarrow S, p_1 \models F.$$

$$S, p \models \mathbf{G} F \Leftrightarrow \forall i \in \mathbb{N} : S, p_i \models F.$$

$$S, p \models \mathbf{F} F \Leftrightarrow \exists i \in \mathbb{N} : S, p_i \models F.$$

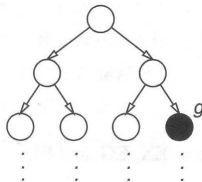
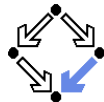
$$S, p \models F \mathbf{U} G \Leftrightarrow \exists i \in \mathbb{N} : S, p_i \models G \wedge \forall j \in \mathbb{N}_i : S, p_j \models F.$$

# Path Formulas

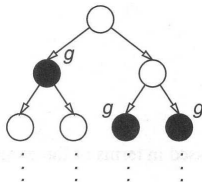


Thomas Kropf: "Introduction to Formal Hardware Verification", 1999.

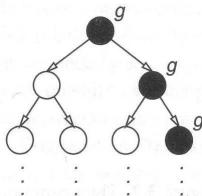
# Path Quantifiers and Temporal Operators



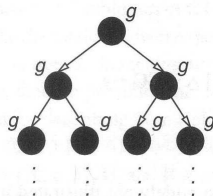
$M, s_0 \models \mathbf{EF} g$



$M, s_0 \models \mathbf{AF} g$

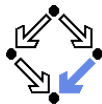


$M, s_0 \models \mathbf{EG} g$



$M, s_0 \models \mathbf{AG} g$

Edmund Clarke et al: "Model Checking", 1999.



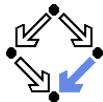
# Linear Time Logic (LTL)

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We use temporal logic to specify a system property  $P$ .

- **Core question:**  $S \models P$  (“ $P$  holds in system  $S$ ”).
  - System  $S = \langle I, R \rangle$ , temporal logic formula  $P$ .
- **Linear time logic:**
  - $S \models P \Leftrightarrow r \models P$ , for every run  $r$  of  $S$ .
  - Property  $P$  must be evaluated on every run  $r$  of  $S$ .
  - Given a computation tree with root  $s_0$ ,  $P$  is evaluated on **every path** of that tree originating in  $s_0$ .
    - If  $P$  holds for every path,  $P$  holds on  $S$ .

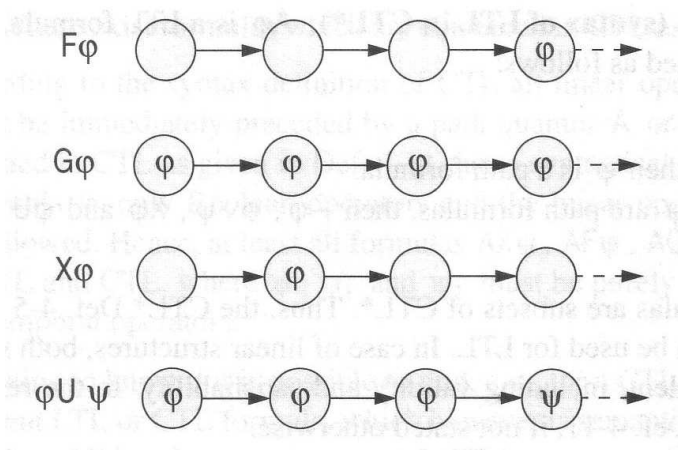
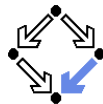
**LTL formulas are evaluated on system runs.**



No path quantifiers; all formulas are path formulas.

- Every **formula** is evaluated on a path  $p$ .
  - Also every state formula  $f$  of classical logic (see below).
  - Let  $F$  and  $G$  denote formulas.
  - Then also the following are formulas:
    - $\mathbf{X} F$  ("next time  $F$ "), often written  $\bigcirc F$ ,
    - $\mathbf{G} F$  ("always  $F$ "), often written  $\square F$ ,
    - $\mathbf{F} F$  ("eventually  $F$ "), often written  $\diamond F$ ,
    - $F \mathbf{U} G$  (" $F$  until  $G$ ").
- **Semantics:**  $p \models P$  (" $P$  holds in path  $p$ ").
  - $p^i := \langle p_i, p_{i+1}, \dots \rangle$ .
  - $p \models f \Leftrightarrow p_0 \models f$ .
  - $p \models \mathbf{X} F \Leftrightarrow p^1 \models F$ .
  - $p \models \mathbf{G} F \Leftrightarrow \forall i \in \mathbb{N} : p^i \models F$ .
  - $p \models \mathbf{F} F \Leftrightarrow \exists i \in \mathbb{N} : p^i \models F$ .
  - $p \models F \mathbf{U} G \Leftrightarrow \exists i \in \mathbb{N} : p^i \models G \wedge \forall j \in \mathbb{N}_i : p^j \models F$ .

# Formulas



Thomas Kropf: "Introduction to Formal Hardware Verification", 1999.

# Branching versus Linear Time Logic



We use temporal logic to specify a system property  $P$ .

- **Core question:**  $S \models P$  (“ $P$  holds in system  $S$ ”).
  - System  $S = \langle I, R \rangle$ , temporal logic formula  $P$ .
- **Branching time logic:**
  - $S \models P \Leftrightarrow S, s_0 \models P$ , for every initial state  $s_0$  of  $S$ .
  - Property  $P$  must be evaluated on every pair  $(S, s_0)$  of system  $S$  and initial state  $s_0$ .
  - Given a computation tree with root  $s_0$ ,  $P$  is evaluated on **that tree**.
- **Linear time logic:**
  - $S \models P \Leftrightarrow r \models P$ , for every run  $r$  of  $s$ .
  - Property  $P$  must be evaluated on every run  $r$  of  $S$ .
  - Given a computation tree with root  $s_0$ ,  $P$  is evaluated on **every path** of that tree originating in  $s_0$ .
    - If  $P$  holds for every path,  $P$  holds on  $S$ .

# Branching versus Linear Time Logic

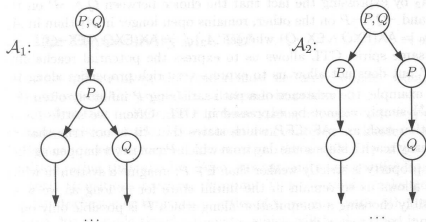
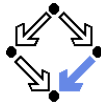


Fig. 2.4. Two automata, indistinguishable for PLTL

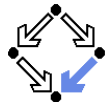
B. Berard et al: "Systems and Software Verification", 2001.

- **Linear time logic:** both systems have the same runs.
  - Thus every formula has same truth value in both systems.
- **Branching time logic:** the systems have different computation trees.
  - Take formula  $\mathbf{AX}(\mathbf{EX} Q \wedge \mathbf{EX} \neg Q)$ .
  - True for left system, false for right system.

The two variants of temporal logic have different expressive power.



# Branching versus Linear Time Logic



Is one temporal logic variant more expressive than the other one?

- CTL formula: **AG(EF F)**.
  - “In every run, it is at any time still **possible** that later  $F$  will hold”.
  - Property cannot be expressed by **any** LTL logic formula.
- LTL formula:  $\diamond\Box F$  (i.e. **FG F**).
  - “In every run, there is a moment from which on  $F$  holds forever.”.
  - Naive translation **AFG F** is **not** a CTL formula.
    - **G F** is a path formula, but **F** expects a state formula!
  - Translation **AFAG F** expresses a **stronger** property (see next page).
  - Property cannot be expressed by **any** CTL formula.

None of the two variants is strictly more expressive than the other one; no variant can express every system property.

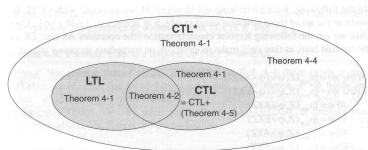
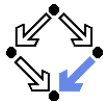


Fig. 4-8. Expressiveness of CTL\*, CTL+, CTL and LTL

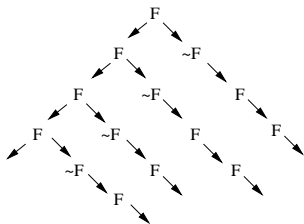
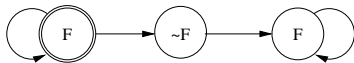
: Thomas Kropf: “Introduction to Formal Hardware Verification”, 1999.

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# Branching versus Linear Time Logic

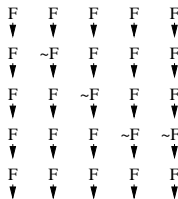


Proof that **AFAG F** (CTL) is different from  $\diamond\Box F$  (LTL).



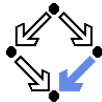
**AFAG F**  $\Leftrightarrow$  false

In every run, there is a moment when it is guaranteed that from now on F holds forever.



$\diamond\Box F$   $\Leftrightarrow$  true

In every run, there is a moment from which on F holds forever.

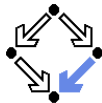


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## 1. The Basics of Temporal Logic

## 2. Specifying with Linear Time Logic

# Linear Time Logic



Why using linear time logic (LTL) for system specifications?

- LTL has many **advantages**:
  - LTL formulas are **easier to understand**.
    - Reasoning about computation paths, not computation trees.
    - No explicit path quantifiers used.
  - LTL can express most interesting system properties.
    - Invariance, guarantee, response, . . . (see later).
  - LTL can express **fairness constraints** (see later).
    - CTL cannot do this.
    - But CTL can express that a state is reachable (which LTL cannot).
- LTL has also some **disadvantages**:
  - LTL is strictly less expressive than other specification languages.
    - CTL\* or  $\mu$ -calculus.
  - Asymptotic complexity of model checking is higher.
    - LTL: exponential in size of formula; CTL: linear in size of formula.
    - In practice the **number of states** dominates the checking time.

# Frequently Used LTL Patterns

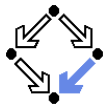


In practice, most temporal formulas are instances of particular patterns.

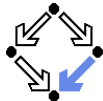
Pattern	Pronounced	Name
$\Box F$	always $F$	invariance
$\Diamond F$	eventually $F$	guarantee
$\Box \Diamond F$	$F$ holds infinitely often	recurrence
$\Diamond \Box F$	eventually $F$ holds permanently	stability
$\Box (F \Rightarrow \Diamond G)$	always, if $F$ holds, then eventually $G$ holds	response
$\Box (F \Rightarrow (G \mathbf{U} H))$	always, if $F$ holds, then $G$ holds until $H$ holds	precedence

Typically, there are at most two levels of nesting of temporal operators.

# Examples



- **Mutual exclusion:**  $\Box \neg (pc_1 = C \wedge pc_2 = C)$ .
  - Alternatively:  $\neg \Diamond (pc_1 = C \wedge pc_2 = C)$ .
  - Never both components are simultaneously in the critical region.
- **No starvation:**  $\forall i : \Box (pc_i = W \Rightarrow \Diamond pc_i = R)$ .
  - Always, if component  $i$  waits for a response, it eventually receives it.
- **No deadlock:**  $\Box \neg \forall i : pc_i = W$ .
  - Never all components are simultaneously in a wait state  $W$ .
- **Precedence:**  $\forall i : \Box (pc_i \neq C \Rightarrow (pc_i \neq C \mathbf{U} lock = i))$ .
  - Always, if component  $i$  is out of the critical region, it stays out until it receives the shared lock variable (which it eventually does).
- **Partial correctness:**  $\Box (pc = L \Rightarrow C)$ .
  - Always if the program reaches line  $L$ , the condition  $C$  holds.
- **Termination:**  $\forall i : \Diamond (pc_i = T)$ .
  - Every component eventually terminates.



# Temporal Rules

Temporal operators obey a number of fairly intuitive rules.

## ■ Extraction laws:

- $\Box F \Leftrightarrow F \wedge \Box F.$
- $\Diamond F \Leftrightarrow F \vee \Diamond F.$
- $F \mathbf{U} G \Leftrightarrow G \vee (F \wedge \Box(F \mathbf{U} G)).$

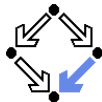
## ■ Negation laws:

- $\neg \Box F \Leftrightarrow \Diamond \neg F.$
- $\neg \Diamond F \Leftrightarrow \Box \neg F.$
- $\neg(F \mathbf{U} G) \Leftrightarrow (\neg G) \mathbf{U} (\neg F \wedge \neg G).$

## ■ Distributivity laws:

- $\Box(F \wedge G) \Leftrightarrow (\Box F) \wedge (\Box G).$
- $\Diamond(F \vee G) \Leftrightarrow (\Diamond F) \vee (\Diamond G).$
- $(F \wedge G) \mathbf{U} H \Leftrightarrow (F \mathbf{U} H) \wedge (G \mathbf{U} H).$
- $F \mathbf{U} (G \vee H) \Leftrightarrow (F \mathbf{U} G) \vee (F \mathbf{U} H).$
- $\Box \Diamond(F \vee G) \Leftrightarrow (\Box \Diamond F) \vee (\Box \Diamond G).$
- $\Diamond \Box(F \wedge G) \Leftrightarrow (\Diamond \Box F) \wedge (\Diamond \Box G).$

# Classes of System Properties



There exists two important classes of system properties.

## ■ Safety Properties:

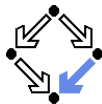
- A safety property is a property such that, if it is violated by a run, it is already violated by some **finite prefix** of the run.
  - This finite prefix cannot be extended in any way to a complete run satisfying the property.
- Example:  $\Box F$ .
  - The violating run  $F \rightarrow F \rightarrow \neg F \rightarrow \dots$  has the prefix  $F \rightarrow F \rightarrow \neg F$  that cannot be extended in any way to a run satisfying  $\Box F$ .

## ■ Liveness Properties:

- A liveness property is a property such that every finite prefix can be extended to a complete run satisfying this property.
  - Only a **complete run itself** can violate that property.
- Example:  $\Diamond F$ .
  - Any finite prefix  $p$  can be extended to a run  $p \rightarrow F \rightarrow \dots$  which satisfies  $\Diamond F$ .



# System Properties



Not every system property is itself a safety property or a liveness property.

- **Example:**  $P :\Leftrightarrow (\Box A) \wedge (\Diamond B)$ 
  - Conjunction of a safety property and a liveness property.
- Take the run  $[A, \neg B] \rightarrow [A, \neg B] \rightarrow [A, \neg B] \rightarrow \dots$  violating  $P$ .
  - Any prefix  $[A, \neg B] \rightarrow \dots \rightarrow [A, \neg B]$  of this run can be extended to a run  $[A, \neg B] \rightarrow \dots \rightarrow [A, \neg B] \rightarrow [A, B] \rightarrow [A, B] \rightarrow \dots$  satisfying  $P$ .
  - Thus  $P$  is **not a safety property**.
- Take the finite prefix  $[\neg A, B]$ .
  - This prefix cannot be extended in any way to a run satisfying  $P$ .
  - Thus  $P$  is **not a liveness property**.

So is the distinction “safety” versus “liveness” really useful?

# System Properties



The real importance of the distinction is stated by the following theorem.

## ■ Theorem:

Every system property  $P$  is a conjunction  $S \wedge L$  of some safety property  $S$  and some liveness property  $L$ .

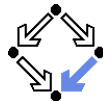
- If  $L$  is “true”, then  $P$  itself is a safety property.
- If  $S$  is “true”, then  $P$  itself is a liveness property.

## ■ Consequence:

- Assume we can decompose  $P$  into appropriate  $S$  and  $L$ .
- For proving  $M \models P$ , it then suffices to perform two proofs:
  - A safety proof:  $M \models S$ .
  - A liveness proof:  $M \models L$ .
- Different strategies for proving safety and liveness properties.

For verification, it is important to decompose a system property in its “safety part” and its “liveness part”.

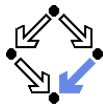
# Proving Invariance



We only consider a special case of a safety property.

- Prove  $M \models \Box F$ .
  - $F$  is a state formula (a formula without temporal operator).
  - Prove that  $F$  is an **invariant** of system  $M$ .
- $M = \langle I, R \rangle$ .
  - $I(s) :\Leftrightarrow \dots$
  - $R(s, s') :\Leftrightarrow R_0(s, s') \vee R_1(s, s') \vee \dots \vee R_{n-1}(s, s')$ .
- **Induction Proof.**
  - $\forall s : I(s) \Rightarrow F(s)$ .
    - Proof that  $F$  holds in every initial state.
  - $\forall s, s' : F(s) \wedge R(s, s') \Rightarrow F(s')$ .
    - Proof that each transition preserves  $F$ .
    - Reduces to a number of subproofs:
      - $F(s) \wedge R_0(s, s') \Rightarrow F(s')$
      - $\dots$
      - $F(s) \wedge R_{n-1}(s, s') \Rightarrow F(s')$

# Example


$$\begin{array}{l} \text{var } x := 0 \\ \text{loop} \\ \quad p_0 : \text{wait } x = 0 \\ \quad p_1 : x := x + 1 \end{array} \quad || \quad \begin{array}{l} \text{loop} \\ \quad q_0 : \text{wait } x = 1 \\ \quad q_1 : x := x - 1 \end{array}$$

$State = \{p_0, p_1\} \times \{q_0, q_1\} \times \mathbb{Z}.$

$I(p, q, x) :\Leftrightarrow p = p_0 \wedge q = q_0 \wedge x = 0.$

$R(\langle p, q, x \rangle, \langle p', q', x' \rangle) :\Leftrightarrow P_0(\dots) \vee P_1(\dots) \vee Q_0(\dots) \vee Q_1(\dots).$

$P_0(\langle p, q, x \rangle, \langle p', q', x' \rangle) :\Leftrightarrow p = p_0 \wedge x = 0 \wedge p' = p_1 \wedge q' = q \wedge x' = x.$

$P_1(\langle p, q, x \rangle, \langle p', q', x' \rangle) :\Leftrightarrow p = p_1 \wedge p' = p_0 \wedge q' = q \wedge x' = x + 1.$

$Q_0(\langle p, q, x \rangle, \langle p', q', x' \rangle) :\Leftrightarrow q = q_0 \wedge x = 1 \wedge p' = p \wedge q' = q_1 \wedge x' = x.$

$Q_1(\langle p, q, x \rangle, \langle p', q', x' \rangle) :\Leftrightarrow q = q_1 \wedge p' = p \wedge q' = q_0 \wedge x' = x - 1.$

Prove  $\langle I, R \rangle \models \square(x = 0 \vee x = 1).$

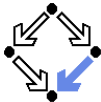
# Inductive System Properties



The induction strategy may not work for proving  $\square F$

- **Problem:**  $F$  is **not inductive**.
  - $F$  is too weak to prove the induction step.
    - $F(s) \wedge R(s, s') \Rightarrow F(s')$ .
- **Solution:** find **stronger** invariant  $I$ .
  - If  $I \Rightarrow F$ , then  $(\square I) \Rightarrow (\square F)$ .
  - It thus suffices to prove  $\square I$ .
- **Rationale:**  $I$  may be **inductive**.
  - If yes,  $I$  is strong enough to prove the induction step.
    - $I(s) \wedge R(s, s') \Rightarrow I(s')$ .
  - If not, find a stronger invariant  $I'$  and try again.
- Invariant  $I$  represents additional knowledge for every proof.
  - Rather than proving  $\square P$ , prove  $\square(I \Rightarrow P)$ .

**The behavior of a system is captured by its strongest invariant.**



## Example

- Prove  $\langle I, R \rangle \models \Box(x = 0 \vee x = 1)$ .
  - Proof attempt fails.
- Prove  $\langle I, R \rangle \models \Box G$ .

$G :\Leftrightarrow$

$$(x = 0 \vee x = 1) \wedge \\ (p = p_1 \Rightarrow x = 0) \wedge \\ (q = q_1 \Rightarrow x = 1).$$

- Proof works.
- $G \Rightarrow (x = 0 \vee x = 1)$  obvious.

See the proof presented in class.

# Proving Liveness


$$\begin{array}{l} \mathbf{var} \ x := 0, y := 0 \\ \mathbf{loop} \\ \quad x := x + 1 \end{array} \quad || \quad \begin{array}{l} \mathbf{loop} \\ \quad y := y + 1 \end{array}$$

$State = \mathbb{N} \times \mathbb{N}; Label = \{p, q\}.$

$I(x, y) :\Leftrightarrow x = 0 \wedge y = 0.$

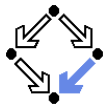
$R(I, \langle x, y \rangle, \langle x', y' \rangle) :\Leftrightarrow$

$(I = p \wedge x' = x + 1 \wedge y' = y) \vee (I = q \wedge x' = x \wedge y' = y + 1).$

- Prove  $\langle I, R \rangle \models \diamond x = 1.$ 
  - $[x = 0, y = 0] \rightarrow [x = 0, y = 1] \rightarrow [x = 0, y = 2] \rightarrow \dots$
  - This run violates (as the only one)  $\diamond x = 1.$
  - Thus the system as a whole does not satisfy  $\diamond x = 1.$

For proving liveness properties, “unfair” runs have to be ruled out.

# Enabling Condition



When is a particular transition enabled for execution?

- $Enabled_R(l, s) :\Leftrightarrow \exists t : R(l, s, t)$ .
  - Labeled transition relation  $R$ , label  $l$ , state  $s$ .
  - Read: “Transition (with label)  $l$  is enabled in state  $s$  (w.r.t.  $R$ )”.
- Example (previous slide):

$Enabled_R(p, \langle x, y \rangle)$

$\Leftrightarrow \exists x', y' : R(p, \langle x, y \rangle, \langle x', y' \rangle)$

$\Leftrightarrow \exists x', y' :$

$(p = p \wedge x' = x + 1 \wedge y' = y) \vee$

$(p = q \wedge x' = x \wedge y' = y + 1)$

$\Leftrightarrow (\exists x', y' : p = p \wedge x' = x + 1 \wedge y' = y) \vee$

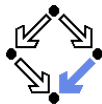
$(\exists x', y' : p = q \wedge x' = x \wedge y' = y + 1)$

$\Leftrightarrow \text{true} \vee \text{false}$

$\Leftrightarrow \text{true}$ .

- Transition  $p$  is always enabled.





# Weak Fairness

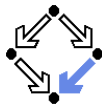
## Weak Fairness

- A run  $s_0 \xrightarrow{l_0} s_1 \xrightarrow{l_1} s_2 \xrightarrow{l_2} \dots$  is **weakly fair** to a transition  $l$ , if
  - if transition  $l$  is eventually **permanently** enabled in the run,
  - then transition  $l$  is executed infinitely often in the run.

$$(\exists i : \forall j \geq i : Enabled_R(l, s_j)) \Rightarrow (\forall i : \exists j \geq i : l_j = l).$$

- The run in the previous example was not weakly fair to transition  $p$ .
- LTL formulas may **explicitly specify** weak fairness constraints.
  - Let  $E_l$  denote the enabling condition of transition  $l$ .
  - Let  $X_l$  denote the predicate “transition  $l$  is executed”.
  - Define  $WF_l : \Leftrightarrow (\diamond \square E_l) \Rightarrow (\square \diamond X_l)$ .
    - If  $l$  is eventually enabled forever, it is executed infinitely often.
  - Prove  $\langle l, S \rangle \models (WF_l \Rightarrow P)$ .
    - Property  $P$  is only proved for runs that are weakly fair to  $l$ .

Alternatively, a model may also have weak fairness “built in”.



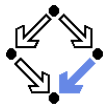
# Proving a Guarantee

We only consider a special case of a liveness property.

- Prove  $\langle I, R \rangle \models \diamond F$ .
  - Proof that  $F$  is a **guarantee** of the system.
  - $F$  is a state formula (a formula without a temporal operator).
- **Decomposition:** sequence of properties  $F_0, F_1, \dots, F_n = F$ .
  - Prove  $\langle I, R \rangle \models \diamond F_0$ .
  - Prove  $\langle I, R \rangle \models \square(F_0 \Rightarrow \diamond F_1)$ .
  - Prove  $\langle I, R \rangle \models \square(F_1 \Rightarrow \diamond F_2)$ .
  - ...
  - Prove  $\langle I, R \rangle \models \square(F_{n-1} \Rightarrow \diamond F)$ .

Typically, guarantee proofs have to be decomposed into multiple proofs.

# Proving a Guarantee



- **Core proof:**  $\langle I, R \rangle \models \diamond F$ .

- Find **lucky transition**  $I$  with enabling condition  $E_I$ .

- The execution of  $I$  makes  $F$  true.
- As long as  $F$  is not true,  $I$  is enabled.
- By weak fairness, either  $F$  becomes true or  $I$  is eventually executed.
- Until  $I$  is executed, additional property  $H$  holds.

$$\neg F(s) \wedge I(s) \Rightarrow H(s) \wedge E_I(s).$$

$$\neg F(s) \wedge H(s) \wedge E_I(s) \wedge \neg R(I, s, s') \Rightarrow H(s') \wedge E_I(s').$$

$$\neg F(s) \wedge H(s) \wedge R(I, s, s') \Rightarrow F(s').$$

- **Core proofs:**  $\langle I, R \rangle \models \square(G \Rightarrow \diamond F)$ .

- Find **lucky transition**  $I$  with enabling condition  $E_I$ .

- Prove:  $\neg F(s) \wedge G(s) \Rightarrow H(s) \wedge E_I(s)$ .
- Prove:  $\neg F(s) \wedge H(s) \wedge E_I(s) \wedge \neg R(I, s, s') \Rightarrow H(s') \wedge E_I(s')$ .
- Prove:  $\neg G(s) \wedge H(s) \wedge R(I, s, s') \Rightarrow F(s')$ .

Sometimes augmented by proofs using well-founded orderings.



# Example

$State = \mathbb{N} \times \mathbb{N}; Label = \{p, q\}.$

$I(x, y) :\Leftrightarrow x = 0 \wedge y = 0.$

$R(I, \langle x, y \rangle, \langle x', y' \rangle) :\Leftrightarrow$

$(I = p \wedge x' = x + 1 \wedge y' = y) \vee (I = q \wedge x' = x \wedge y' = y + 1).$

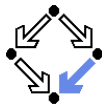
■ Prove  $\langle I, R \rangle \models \Diamond x = 1.$

■ Lucky transition  $p$ , additional property  $H :\Leftrightarrow x = 0.$

$x \neq 1 \wedge (x = 0 \wedge y = 0) \Rightarrow x = 0 \wedge \text{true}.$

$x \neq 1 \wedge x = 0 \wedge \text{true} \wedge (x' = x \wedge y' = y + 1) \Rightarrow x' = 0 \wedge \text{true}.$

$x \neq 1 \wedge x = 0 \wedge (x' = x + 1 \wedge y' = y) \Rightarrow x' = 1.$



# Strong Fairness

## ■ Strong Fairness

- A run  $s_0 \xrightarrow{l_0} s_1 \xrightarrow{l_1} s_2 \xrightarrow{l_2} \dots$  is **strongly fair** to a transition  $l$ , if
  - if  $l$  is **infinitely often** enabled in the run,
  - then  $l$  is also infinitely often executed the run.

$$(\forall i : \exists j \geq i : Enabled_R(l, s_j)) \Rightarrow (\forall i : \exists j \geq i : l_j = l).$$

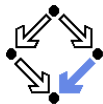
- If  $r$  is strongly fair to  $l$ , it is also weakly fair to  $l$  (but not vice versa).
- LTL formulas may **explicitly specify** strong fairness constraints.
  - Let  $E_l$  denote the enabling condition of transition  $l$ .
  - Let  $X_l$  denote the predicate “transition  $l$  is executed”.
  - Define  $SF_l : \Leftrightarrow (\Box \Diamond E_l) \Rightarrow (\Box \Diamond X_l)$ .

If  $l$  is enabled infinitely often, it is executed infinitely often.
  - Prove  $\langle l, S \rangle \models (SF_l \Rightarrow P)$ .

Property  $P$  is only proved for runs that are strongly fair to  $l$ .

**A much stronger requirement to the fairness of a system.**

# Example



```
var x=0
loop
  a : x := -x
  b : choose x := 0 [] x := 1
```

$State := \{a, b\} \times \mathbb{Z}; Label = \{A, B_0, B_1\}.$

$I(p, x) :\Leftrightarrow p = a \wedge x = 0.$

$R(I, \langle p, x \rangle, \langle p', x' \rangle) :\Leftrightarrow$

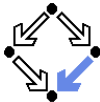
$(I = A \wedge (p = a \wedge p' = b \wedge x' = -x)) \vee$

$(I = B_0 \wedge (p = b \wedge p' = a \wedge x' = 0)) \vee$

$(I = B_1 \wedge (p = b \wedge p' = a \wedge x' = 1)).$

■ Prove:  $\langle I, R \rangle \models \diamond x = 1.$

- Take violating run  $[a, 0] \xrightarrow{A} [b, 0] \xrightarrow{B_0} [a, 0] \xrightarrow{A} [b, 0] \xrightarrow{B_0} [a, 0] \xrightarrow{A} \dots$
- $Enabled_{B_1}(p, x) :\Leftrightarrow p = b.$
- Run is weakly fair **but not strongly fair** to  $B_1.$



# Weak versus Strong Fairness

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In which situations is which notion of fairness appropriate?

- Process just waits to be scheduled for execution.
  - Only CPU time is required.
  - Weak fairness suffices.
- Process waits for resource that may be temporarily blocked.
  - Critical region protected by lock variable (mutex/semaphore).
  - Strong fairness is required.
- Non-deterministic choices are repeatedly made in program.
  - Simultaneous listing on multiple communication channels.
  - Strong fairness is required.

Many other notions of fairness exist.