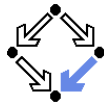


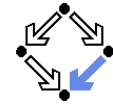
The Language of Logic

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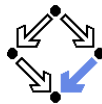
The Language of Logic



Two kinds of syntactic phrases.

- **Term** T denoting an **object**.
 - Variable x
 - Object constant c
 - Function application $f(T_1, \dots, T_n)$
 n -ary function constant f (may be written infix)
- **Formula** F denoting a **truth value**.
 - Atomic formula $p(T_1, \dots, T_n)$ (may be written infix)
 n -ary predicate constant p .
 - Negation $\neg F$ ("not F ")
 - Conjunction $F_1 \wedge F_2$ (" F_1 and F_2 ")
 - Disjunction $F_1 \vee F_2$ (" F_1 or F_2 ")
 - Implication $F_1 \Rightarrow F_2$ ("if F_1 , then F_2 ")
 - Equivalence $F_1 \Leftrightarrow F_2$ ("if F_1 , then F_2 , and vice versa")
 - Universal quantification $\forall x : F$ ("for all x , F ")
 - Existential quantification $\exists x : F$ ("for some x , F ")

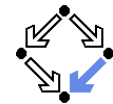
Syntactic Shortcuts



- $\forall x_1, \dots, x_n : F$
 - $\forall x_1 : \dots : \forall x_n : F$
- $\exists x_1, \dots, x_n : F$
 - $\exists x_1 : \dots : \exists x_n : F$
- $\forall x \in S : F$
 - $\forall x : x \in S \Rightarrow F$
- $\exists x \in S : F$
 - $\exists x : x \in S \wedge F$

Help to make formulas more readable.

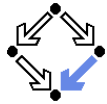
Examples



Terms and formulas may appear in various syntactic forms.

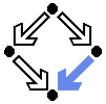
- **Terms:**
 - $\exp(x)$
 - $a \cdot b + 1$
 - $a[i] \cdot b$
 - $\sqrt{\frac{x^2 + 2x + 1}{(y+1)^2}}$
- **Formulas:**
 - $a^2 + b^2 = c^2$
 - $n \mid 2n$
 - $\forall x \in \mathbb{N} : x \geq 0$
 - $\forall x \in \mathbb{N} : 2 \mid x \vee 2 \mid (x + 1)$
 - $\forall x \in \mathbb{N}, y \in \mathbb{N} : x < y \Rightarrow$
 - $\exists z \in \mathbb{N} : x + z = y$

Terms and formulas may be nested arbitrarily deeply.



The Meaning of Formulas

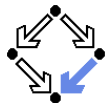
- Atomic formula $p(T_1, \dots, T_n)$
 - True if the predicate denoted by p holds for the values of T_1, \dots, T_n .
- Negation $\neg F$
 - True if and only if F is false.
- Conjunction $F_1 \wedge F_2$ (“ F_1 and F_2 ”)
 - True if and only if F_1 and F_2 are both true.
- Disjunction $F_1 \vee F_2$ (“ F_1 or F_2 ”)
 - True if and only if at least one of F_1 or F_2 is true.
- Implication $F_1 \Rightarrow F_2$ (“if F_1 , then F_2 ”)
 - False if and only if F_1 is true and F_2 is false.
- Equivalence $F_1 \Leftrightarrow F_2$ (“if F_1 , then F_2 , and vice versa”)
 - True if and only if F_1 and F_2 are both true or both false.
- Universal quantification $\forall x : F$ (“for all x , F ”)
 - True if and only if F is true for every possible value assignment of x .
- Existential quantification $\exists x : F$ (“for some x , F ”)
 - True if and only if F is true for at least one value assignment of x .



Example

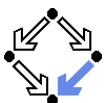
We assume the domain of natural numbers and the “classical” interpretation of constants $1, 2, +, =, <$.

- $1 + 1 = 2$
 - True.
- $1 + 1 = 2 \vee 2 + 2 = 2$
 - True.
- $1 + 1 = 2 \wedge 2 + 2 = 2$
 - False.
- $1 + 1 = 2 \Rightarrow 2 = 1 + 1$
 - True.
- $1 + 1 = 1 \Rightarrow 2 + 2 = 2$
 - True.
- $1 + 1 = 2 \Rightarrow 2 + 2 = 2$
 - False.
- $1 + 1 = 1 \Leftrightarrow 2 + 2 = 2$
 - True.



Example

- $x + 1 = 1 + x$
 - True, for every assignment of a number a to variable x .
- $\forall x : x + 1 = 1 + x$
 - True (because for every assignment a to x , $x + 1 = 1 + x$ is true).
- $x + 1 = 2$
 - If x is assigned “one”, the formula is true.
 - If x is assigned “two”, the formula is false.
- $\exists x : x + 1 = 2$
 - True (because $x + 1 = 2$ is true for assignment “one” to x).
- $\forall x : x + 1 = 2$
 - False (because $x + 1 = 2$ is false for assignment “two” to x).
- $\forall x : \exists y : x < y$
 - True (because for every assignment a to x , there exists the assignment $a + 1$ to y which makes $x < y$ true).
- $\exists y : \forall x : x < y$
 - False (because for every assignment a to y , there is the assignment $a + 1$ to x which makes $x < y$ false).

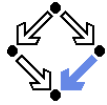


Formula Equivalences

Formulas may be replaced by equivalent formulas.

- $\neg\neg F_1 \Leftrightarrow F_1$
- $\neg(F_1 \wedge F_2) \Leftrightarrow \neg F_1 \vee \neg F_2$
- $\neg(F_1 \vee F_2) \Leftrightarrow \neg F_1 \wedge \neg F_2$
- $\neg(F_1 \Rightarrow F_2) \Leftrightarrow F_1 \wedge \neg F_2$
- $\neg\forall x : F \Leftrightarrow \exists x : \neg F$
- $\neg\exists x : F \Leftrightarrow \forall x : \neg F$
- $F_1 \Rightarrow F_2 \Leftrightarrow \neg F_2 \Rightarrow \neg F_1$
- $F_1 \Rightarrow F_2 \Leftrightarrow \neg F_1 \vee F_2$
- $F_1 \Leftrightarrow F_2 \Leftrightarrow \neg F_1 \Leftrightarrow \neg F_2$
- ...

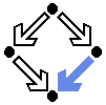
Familiarity with manipulation of formulas is important.



Example

- “All swans are white or black.”
 - $\forall x : \text{swan}(x) \Rightarrow \text{white}(x) \vee \text{black}(x)$
- “There exists a black swan.”
 - $\exists x : \text{swan}(x) \wedge \text{black}(x)$.
- “A swan is white, unless it is black.”
 - $\forall x : \text{swan}(x) \wedge \neg \text{black}(x) \Rightarrow \text{white}(x)$
 - $\forall x : \text{swan}(x) \wedge \neg \text{white}(x) \Rightarrow \text{black}(x)$
 - $\forall x : \text{swan}(x) \Rightarrow \text{white}(x) \vee \text{black}(x)$
- “Not everything that is white or black is a swan.”
 - $\neg \forall x : \text{white}(x) \vee \text{black}(x) \Rightarrow \text{swan}(x)$.
 - $\exists x : \text{white}(x) \wedge \text{black}(x) \wedge \neg \text{swan}(x)$.
- “Black swans have at least one black parent”.
 - $\forall x : \text{swan}(x) \wedge \text{black}(x) \Rightarrow \exists y : \text{swan}(y) \wedge \text{black}(y) \wedge \text{parent}(y, x)$

It is important to recognize the logical structure of an informal sentence in its various equivalent forms.



The Usage of Formulas

Precise formulation of statements describing object relationships.

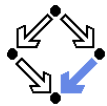
- **Statement:**

If x and y are natural numbers and y is not zero, then q is the truncated quotient of x divided by y .
- **Formula:**

$$x \in \mathbb{N} \wedge y \in \mathbb{N} \wedge y \neq 0 \Rightarrow q \in \mathbb{N} \wedge \exists r \in \mathbb{N} : r < y \wedge x = y \cdot q + r$$
- **Problem specification:**

Given natural numbers x and y such that y is not zero, compute the truncated quotient q of x divided by y .

 - Inputs: x, y
 - Input condition: $x \in \mathbb{N} \wedge y \in \mathbb{N} \wedge y \neq 0$
 - Output: q
 - Output condition: $q \in \mathbb{N} \wedge \exists r \in \mathbb{N} : r < y \wedge x = y \cdot q + r$



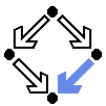
Problem Specifications

- **Specification** of a computation problem:
 - Input: variables $X_1 \in S_1, \dots, X_n \in S_n$
 - Input condition: formula I in which only X_1, \dots, X_n may be free.
 - Output: variables $Y_1 \in T_1, \dots, Y_m \in T_n$
 - Output condition: formula O in which only $X_1, \dots, X_n, Y_1, \dots, Y_m$ may be free.

A variable is **free** in a formula, if it occurs in the formula outside the scope of a quantifier (such as \forall or \exists).
- **Implementation** of the specification:
 - A function (program) $f : S_1 \times \dots \times S_n \rightarrow T_1 \times \dots \times T_m$ such that

$$\forall X_1 \in S_1, \dots, X_n \in S_n : I(X_1, \dots, X_n) \Rightarrow \text{let } (Y_1, \dots, Y_m) = f(X_1, \dots, X_n) \text{ in } O(X_1, \dots, X_n, Y_1, \dots, Y_m)$$
 - For all arguments that satisfy the input condition, f must compute results that satisfy the output condition.

Basis of all specification formalisms.



Example: A Program Specification

Given an integer array a , a position p in a , and a length l , return the array b derived from a by removing $a[p], \dots, a[p + l]$.

- **Input:** $a \in \mathbb{Z}^*, p \in \mathbb{N}, l \in \mathbb{N}$
- **Input condition:**

$$p + l \leq \text{length}_{\mathbb{Z}}(a)$$
- **Output:** $b \in \mathbb{Z}^*$
- **Output condition:**

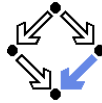
$$\text{let } n = \text{length}_{\mathbb{Z}}(a) \text{ in } \text{length}(b) = n - l \wedge (\forall i \in \mathbb{N} : i < p \Rightarrow b[i] = a[i]) \wedge (\forall i \in \mathbb{N} : p \leq i < n - l \Rightarrow b[i] = a[i + l])$$

Mathematical theory:

$$T^* := \bigcup_{i \in \mathbb{N}} T^i, T^i := \mathbb{N}_i \rightarrow T, \mathbb{N}_i := \{n \in \mathbb{N} : n < i\}$$

$$\text{length}_T : T^* \rightarrow \mathbb{N}, \text{length}_T(a) = \text{such } i \in \mathbb{N} : a \in T^i$$

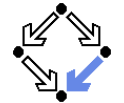
Proving Formulas



- **Proof:** a structured argument that a formula is true.
 - Can be depicted as a tree whose nodes represent proof situations.
 - Each proof situation consists of *knowledge* (formulas assumed to be true) and a *goal* (a formula to be proved relative to that knowledge).
 - The root goal is the overall formula to be proved.
- **Proof rules:** describe how a proof situation can be reduced to zero or more “subsituations”.
 - Zero subsituations: the current goal has been proved.
 - One or more subsituations: the current goal has been proved, if all subgoals have been proved.
 - **Top-down rules:** focus on goal formula.
 - Goal formula is decomposed into simpler formulas.
 - **Bottom-up rules:** focus on some formula in knowledge.
 - Additional formulas are added to the knowledge.

In each proof situation, we aim at showing that the goal is “apparently” true with respect to the given knowledge.

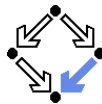
Conjunction



Formula $F_1 \wedge F_2$.

- **Formula as goal.**
 - Create two subsituations with goals F_1 and F_2 .
We have to show $F_1 \wedge F_2$.
 - *We show F_1 : ... (proof continues with goal F_1)*
 - *We show F_2 : ... (proof continues with goal F_2)*
- **Formula in knowledge.**
 - Create one subsituation with F_1 and F_2 in knowledge.
We know $F_1 \wedge F_2$. We thus know F_1 and also F_2 . (proof continues with current goal and additional knowledge F_1 and F_2)

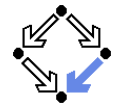
Disjunction



Formula $F_1 \vee F_2$.

- **Formula as goal.**
 - Create one subsituation where F_2 is proved under the assumption that F_1 does not hold (or vice versa):
We have to show $F_1 \vee F_2$. We assume $\neg F_1$ and show F_2 . (proof continues with goal F_2 and additional knowledge $\neg F_1$)
- **Formula in knowledge.**
 - Create two subsituations, one with F_1 and one with F_2 in knowledge.
We know $F_1 \vee F_2$. We thus proceed by case distinction:
 - *Case F_1 : ... (proof continues with current goal and additional knowledge F_1).*
 - *Case F_2 : ... (proof continues with current goal and additional knowledge F_2).*

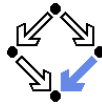
Implication



Formula $F_1 \Rightarrow F_2$.

- **Formula as goal.**
 - Create one subsituation where F_2 is proved under the assumption that F_1 holds:
We have to show $F_1 \Rightarrow F_2$. We assume F_1 and show F_2 . (proof continues with goal F_2 and additional knowledge F_1)
- **Formula in knowledge.**
 - Create two subsituations, one with goal F_1 and one with knowledge F_2 .
We know $F_1 \Rightarrow F_2$.
 - *We show F_1 : ... (proof continues with goal F_1)*
 - *We know F_2 : ... (proof continues with current goal and additional knowledge F_2).*

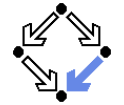
Equivalence



Formula $F_1 \Leftrightarrow F_2$.

- **Formula as goal.**
 - Create two subsituations with implications in both directions as goals:
We have to show $F_1 \Leftrightarrow F_2$.
 - *We show $F_1 \Rightarrow F_2$: ... (proof continues with goal $F_1 \Rightarrow F_2$)*
 - *We show $F_2 \Rightarrow F_1$: ... (proof continues with goal $F_2 \Rightarrow F_1$)*
- **Formula in knowledge.**
 - Create two subsituations, one with goal F_1 and one with knowledge F_2 .
We know $F_1 \Leftrightarrow F_2$.
 - *We show $F_1 (\neg F_1)$: ... (proof continues with goal F_1)*
 - *We know $F_2 (\neg F_2)$: ... (proof continues with current goal and additional knowledge F_2)*

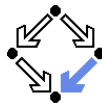
Universal Quantification



Formula $\forall x : F$

- **Formula as goal.**
 - Introduce new (arbitrarily named) constant x_0 and create one subsituation with goal $F[x_0/x]$.
We have to show $\forall x : F$. Take arbitrary x_0 .
We show $F[x_0/x]$. (proof continues with goal $F[x_0/x]$)
- **Formula in knowledge.**
 - Choose term T to create one subsituation with formula $F[T/x]$ added to the knowledge.
We know $\forall x : F$ and thus also $F[T/x]$.
(proof continues with current goal and additional knowledge $F[T/x]$)

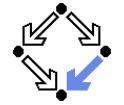
Existential Quantification



Formula $\exists x : F$

- **Formula as goal.**
 - Choose term T to create one subsituation with goal $F[T/x]$.
We have to show $\exists x : F$. It suffices to show $F[T/x]$.
(proof continues with goal $F[T/x]$)
- **Formula in knowledge.**
 - Introduce new (arbitrarily named constant) x_0 and create one subsituation with additional knowledge $F[x_0/x]$.
We know $\exists x : F$. Let x_0 be such that $F[x_0/x]$.
(proof continues with current goal and additional knowledge $F[x_0/x]$)

Example



We show

$$(a) (\exists x : \forall y : P(x, y)) \Rightarrow (\forall y : \exists x : P(x, y))$$

We assume

$$(1) \exists x : \forall y : P(x, y)$$

and show

$$(b) \forall y : \exists x : P(x, y)$$

Take arbitrary y_0 . We show

$$(c) \exists x : P(x, y_0)$$

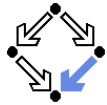
From (1) we know for some x_0

$$(2) \forall y : P(x_0, y)$$

From (2) we know

$$(3) P(x_0, y_0)$$

From (3), we know (c). QED.



Example

We show

$$(a) (\exists x : p(x)) \wedge (\forall x : p(x) \Rightarrow \exists y : q(x, y)) \Rightarrow (\exists x, y : q(x, y))$$

We assume

$$(1) (\exists x : p(x)) \wedge (\forall x : p(x) \Rightarrow \exists y : q(x, y))$$

and show

$$(b) \exists x, y : q(x, y)$$

From (1), we know

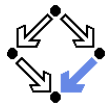
$$(2) \exists x : p(x)$$

$$(3) \forall x : p(x) \Rightarrow \exists y : q(x, y)$$

From (2) we know for some x_0

$$(4) p(x_0)$$

...

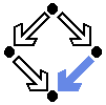


Indirect Proofs

Proof situation with goal formula G .

- Add $\neg G$ to the knowledge and show a contradiction.
 - Prove that “false” is true.
 - Prove that a formula F is true and also prove that it is false.
 - Prove that a formula F in the knowledge is false, i.e. that $\neg F$ is true.
 - Switches goal G and some knowledge F (negating both).

Sometimes simpler than a direct proof.



Example (Contd)

...

From (3), we know

$$(5) p(x_0) \Rightarrow \exists y : q(x_0, y)$$

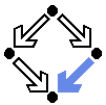
From (4) and (5), we know

$$(6) \exists y : q(x_0, y)$$

From (6), we know for some y_0

$$(7) q(x_0, y_0)$$

From (7), we know (b). QED.



Example

We show

$$(a) (\exists x : \forall y : P(x, y)) \Rightarrow (\forall y : \exists x : P(x, y))$$

We assume

$$(1) \exists x : \forall y : P(x, y)$$

and show

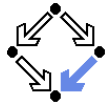
$$(b) \forall y : \exists x : P(x, y)$$

We assume

$$(2) \neg \forall y : \exists x : P(x, y)$$

and show a contradiction.

...



Example

...

From (2), we know

$$(3) \exists y : \forall x : \neg P(x, y)$$

Let y_0 be such that

$$(4) \forall x : \neg P(x, y_0)$$

From (1) we know for some x_0

$$(5) \forall y : P(x_0, y)$$

From (5) we know

$$(6) P(x_0, y_0)$$

From (4), we know

$$(7) \neg P(x_0, y_0)$$

From (6) and (7), we have a contradiction. QED.