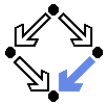


# Specifying Properties of Concurrent Systems

Wolfgang Schreiner  
Wolfgang.Schreiner@risc.uni-linz.ac.at

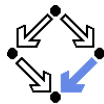
Research Institute for Symbolic Computation (RISC)  
Johannes Kepler University, Linz, Austria  
<http://www.risc.uni-linz.ac.at>



## 1. The Basics of Temporal Logic

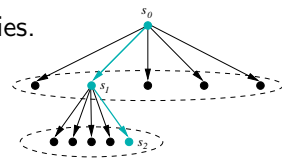
## 2. Specifying with Linear Time Logic

## Motivation



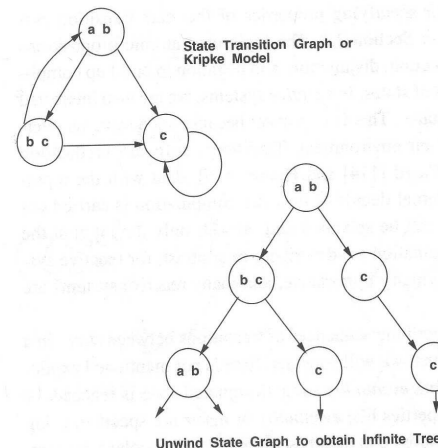
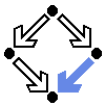
We need a language for specifying system properties.

- A system  $S$  is a pair  $\langle I, R \rangle$ .
  - Initial states  $I$ , transition relation  $R$ .
  - More intuitive: reachability graph.
    - Starting from an initial state  $s_0$ , the system runs evolve.
- Consider the reachability graph as an infinite **computation tree**.
  - Different tree nodes may denote occurrences of the same state.
    - Each occurrence of a state has a unique predecessor in the tree.
  - Every path in this tree is infinite.
    - Every finite run  $s_0 \rightarrow \dots \rightarrow s_n$  is extended to an infinite run  $s_0 \rightarrow \dots \rightarrow s_n \rightarrow s_n \rightarrow s_n \rightarrow \dots$
- Or simply consider the graph as a **set of system runs**.
  - Same state may occur multiple times (in one or in different runs).



Temporal logic describes such trees respectively sets of system runs.

## Computation Trees versus System Runs



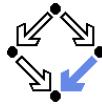
Set of system runs:

- $[a, b] \rightarrow c \rightarrow c \rightarrow \dots$
- $[a, b] \rightarrow [b, c] \rightarrow c \rightarrow \dots$
- $[a, b] \rightarrow [b, c] \rightarrow [a, b] \rightarrow \dots$
- $[a, b] \rightarrow [b, c] \rightarrow [a, b] \rightarrow \dots$
- ...

Figure 3.1  
Computation trees...

Edmund Clarke et al: "Model Checking", 1999.

## State Formula

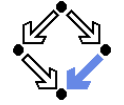


Temporal logic is based on classical logic.

- A **state formula**  $F$  is evaluated on a state  $s$ .
  - Any predicate logic formula is a state formula:  
 $p(x), \neg F, F_0 \wedge F_1, F_0 \vee F_1, F_0 \Rightarrow F_1, F_0 \Leftrightarrow F_1, \forall x : F, \exists x : F$ .
  - In **propositional temporal logic** only propositional logic formulas are state formulas (no quantification):  
 $p, \neg F, F_0 \wedge F_1, F_0 \vee F_1, F_0 \Rightarrow F_1, F_0 \Leftrightarrow F_1$ .
- **Semantics**:  $s \models F$  (“ $F$  holds in state  $s$ ”).
  - Example: semantics of conjunction.
    - $(s \models F_0 \wedge F_1) :\Leftrightarrow (s \models F_0) \wedge (s \models F_1)$ .
    - “ $F_0 \wedge F_1$  holds in  $s$  if and only if  $F_0$  holds in  $s$  and  $F_1$  holds in  $s$ ”.

Classical logic reasons on individual states.

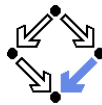
## Temporal Logic



Extension of classical logic to reason about multiple states.

- Temporal logic is an instance of **modal logic**.
  - Logic of “multiple worlds (situations)” that are in some way related.
  - Relationship may e.g. be a **temporal** one.
  - Amir Pnueli, 1977: temporal logic is suited to system specifications.
  - Many variants, two fundamental classes.
- **Branching Time Logic**
  - Semantics defined over **computation trees**.  
At each moment, there are multiple possible futures.
  - Prominent variant: **CTL**.  
Computation tree logic; a propositional branching time logic.
- **Linear Time Logic**
  - Semantics defined over **sets of system runs**.  
At each moment, there is only one possible future.
  - Prominent variant: **PLTL**.  
A propositional linear time logic.

## Branching Time Logic (CTL)

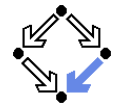


We use temporal logic to specify a system property  $F$ .

- **Core question**:  $S \models F$  (“ $F$  holds in system  $S$ ”).
  - System  $S = \langle I, R \rangle$ , temporal logic formula  $F$ .
- **Branching time logic**:
  - $S \models F :\Leftrightarrow S, s_0 \models F$ , for every initial state  $s_0$  of  $S$ .
  - Property  $F$  must be evaluated on every pair of system  $S$  and initial state  $s_0$ .
  - Given a computation tree with root  $s_0$ ,  $F$  is evaluated on **that tree**.

CTL formulas are evaluated on computation trees.

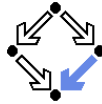
## State Formulas



We have additional state formulas.

- A **state formula**  $F$  is evaluated on state  $s$  of System  $S$ .
  - Every (classical) state formula  $f$  is such a state formula.
  - Let  $P$  denote a **path formula** (later).
    - Evaluated on a **path** (state sequence)  $p = p_0 \rightarrow p_1 \rightarrow p_2 \rightarrow \dots$   
 $R(p_i, p_{i+1})$  for every  $i$ ;  $p_0$  need not be an initial state.
  - Then the following are **state formulas**:
    - **A**  $P$  (“in every path  $P$ ”),
    - **E**  $P$  (“in some path  $P$ ”).
  - **Path quantifiers**: **A, E**.
- **Semantics**:  $S, s \models F$  (“ $F$  holds in state  $s$  of system  $S$ ”).
  - $S, s \models f :\Leftrightarrow s \models f$ .
  - $S, s \models \mathbf{A} P :\Leftrightarrow S, p \models P$ , for every path  $p$  of  $S$  with  $p_0 = s$ .
  - $S, s \models \mathbf{E} P :\Leftrightarrow S, p \models P$ , for some path  $p$  of  $S$  with  $p_0 = s$ .

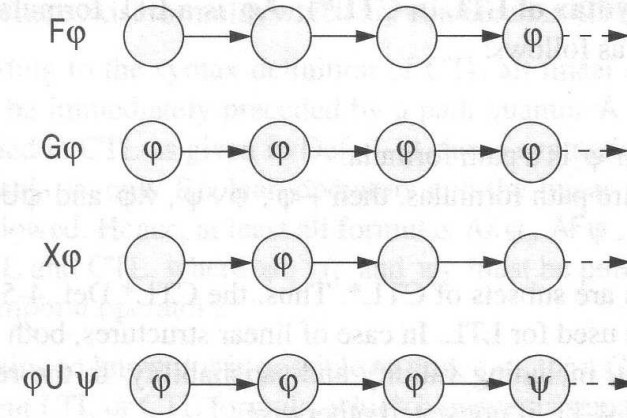
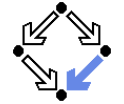
# Path Formulas



We have a class of formulas that are not evaluated over individual states.

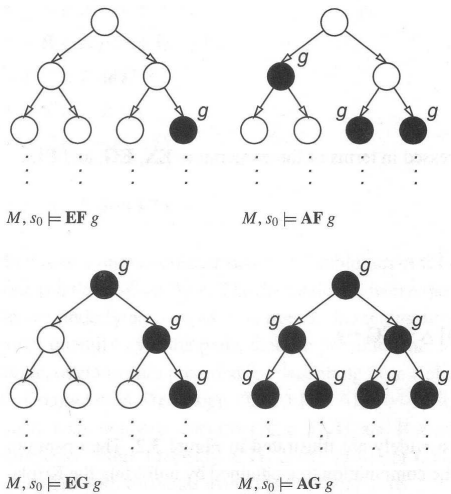
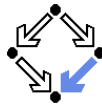
- A **path formula**  $P$  is evaluated on a path  $p$  of system  $S$ .
  - Let  $F$  and  $G$  denote **state formulas**.
  - Then the following are **path formulas**:
    - $\mathbf{X} F$  ("next time  $F$ "),
    - $\mathbf{G} F$  ("always  $F$ "),
    - $\mathbf{F} F$  ("eventually  $F$ "),
    - $F \mathbf{U} G$  (" $F$  until  $G$ ").
  - **Temporal operators:  $\mathbf{X}, \mathbf{G}, \mathbf{F}, \mathbf{U}$ .**
- **Semantics:**  $S, p \models P$  (" $P$  holds in path  $p$  of system  $S$ ").
  - $S, p \models \mathbf{X} F \iff S, p_1 \models F.$
  - $S, p \models \mathbf{G} F \iff \forall i \in \mathbb{N} : S, p_i \models F.$
  - $S, p \models \mathbf{F} F \iff \exists i \in \mathbb{N} : S, p_i \models F.$
  - $S, p \models F \mathbf{U} G \iff \exists i \in \mathbb{N} : S, p_i \models G \wedge \forall j \in \mathbb{N}_i : S, p_j \models F.$

# Path Formulas



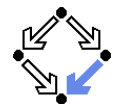
Thomas Kropf: "Introduction to Formal Hardware Verification", 1999.

# Path Quantifiers and Temporal Operators



Edmund Clarke et al: "Model Checking", 1999.

# Linear Time Logic (LTL)

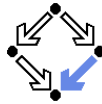


We use temporal logic to specify a system property  $P$ .

- **Core question:**  $S \models P$  (" $P$  holds in system  $S$ ").
  - System  $S = \langle I, R \rangle$ , temporal logic formula  $P$ .
- **Linear time logic:**
  - $S \models P \iff r \models P$ , for every run  $r$  of  $S$ .
  - Property  $P$  must be evaluated on every run  $r$  of  $S$ .
  - Given a computation tree with root  $s_0$ ,  $P$  is evaluated on **every path** of that tree originating in  $s_0$ .
    - If  $P$  holds for every path,  $P$  holds on  $S$ .

**LTL formulas are evaluated on system runs.**

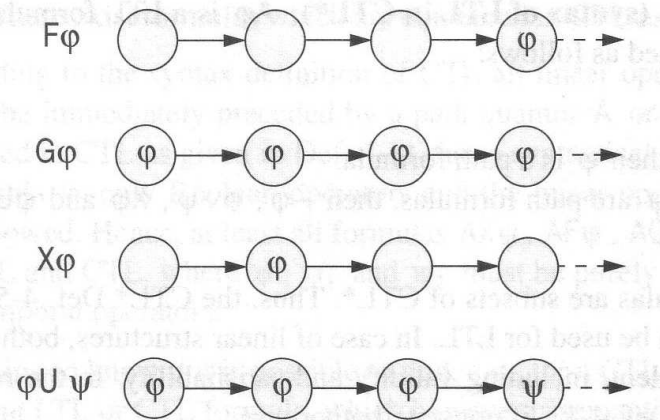
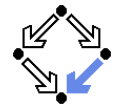
# Formulas



No path quantifiers; all formulas are path formulas.

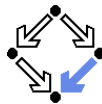
- Every **formula** is evaluated on a path  $p$ .
  - Also every state formula  $f$  of classical logic (see below).
  - Let  $F$  and  $G$  denote formulas.
  - Then also the following are formulas:
    - $\mathbf{X} F$  ("next time  $F$ "), often written  $\circ F$ ,
    - $\mathbf{G} F$  ("always  $F$ "), often written  $\square F$ ,
    - $\mathbf{F} F$  ("eventually  $F$ "), often written  $\diamond F$ ,
    - $F \mathbf{U} G$  (" $F$  until  $G$ ").
- **Semantics:**  $p \models P$  (" $P$  holds in path  $p$ ").
  - $p^i := \langle p_i, p_{i+1}, \dots \rangle$ .
  - $p \models f \Leftrightarrow p_0 \models f$ .
  - $p \models \mathbf{X} F \Leftrightarrow p^1 \models F$ .
  - $p \models \mathbf{G} F \Leftrightarrow \forall i \in \mathbb{N} : p^i \models F$ .
  - $p \models \mathbf{F} F \Leftrightarrow \exists i \in \mathbb{N} : p^i \models F$ .
  - $p \models F \mathbf{U} G \Leftrightarrow \exists i \in \mathbb{N} : p^i \models G \wedge \forall j \in \mathbb{N}_i : p^j \models F$ .

# Formulas



Thomas Kropf: "Introduction to Formal Hardware Verification", 1999.

# Branching versus Linear Time Logic



We use temporal logic to specify a system property  $P$ .

- **Core question:**  $S \models P$  (" $P$  holds in system  $S$ ").
  - System  $S = \langle I, R \rangle$ , temporal logic formula  $P$ .
- **Branching time logic:**
  - $S \models P \Leftrightarrow S, s_0 \models P$ , for every initial state  $s_0$  of  $S$ .
  - Property  $P$  must be evaluated on every pair  $(S, s_0)$  of system  $S$  and initial state  $s_0$ .
  - Given a computation tree with root  $s_0$ ,  $P$  is evaluated on **that tree**.
- **Linear time logic:**
  - $S \models P \Leftrightarrow r \models P$ , for every run  $r$  of  $s$ .
  - Property  $P$  must be evaluated on every run  $r$  of  $S$ .
  - Given a computation tree with root  $s_0$ ,  $P$  is evaluated on **every path** of that tree originating in  $s_0$ .
    - If  $P$  holds for every path,  $P$  holds on  $S$ .

# Branching versus Linear Time Logic

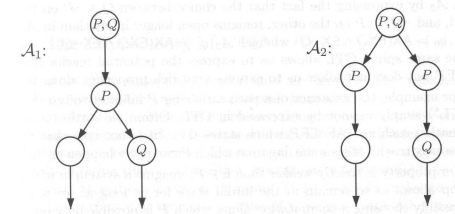
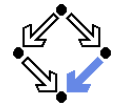


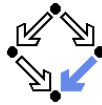
Fig. 2.4. Two automata, indistinguishable for PLTL

B. Berard et al: "Systems and Software Verification", 2001.

- **Linear time logic:** both systems have the same runs.
  - Thus every formula has same truth value in both systems.
- **Branching time logic:** the systems have different computation trees.
  - Take formula  $\mathbf{AX}(\mathbf{EX} Q \wedge \mathbf{EX} \neg Q)$ .
  - True for left system, false for right system.

The two variants of temporal logic have different expressive power.

# Branching versus Linear Time Logic



Is one temporal logic variant more expressive than the other one?

- CTL formula: **AG(EF F)**.
  - "In every run, it is at any time still possible that later F will hold".
  - Property cannot be expressed by any LTL logic formula.
- LTL formula:  $\diamond\Box F$  (i.e. **FG F**).
  - "In every run, there is a moment from which on F holds forever."
  - Naive translation **AFG F** is not a CTL formula.
    - **G F** is a path formula, but **F** expects a state formula!
  - Translation **AFAG F** expresses a stronger property (see next page).
  - Property cannot be expressed by any CTL formula.

None of the two variants is strictly more expressive than the other one; no variant can express every system property.

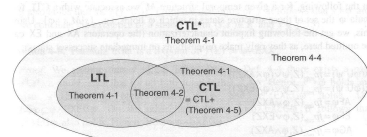


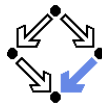
Fig. 4-8. Expressiveness of CTL\*, CTL+, CTL and LTL

: Thomas Kropf: "Introduction to Formal Hardware Verification", 1999.

<http://www.risc.uni-linz.ac.at>

17/39

Wolfgang Schreiner



## 1. The Basics of Temporal Logic

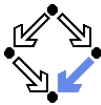
## 2. Specifying with Linear Time Logic

Wolfgang Schreiner

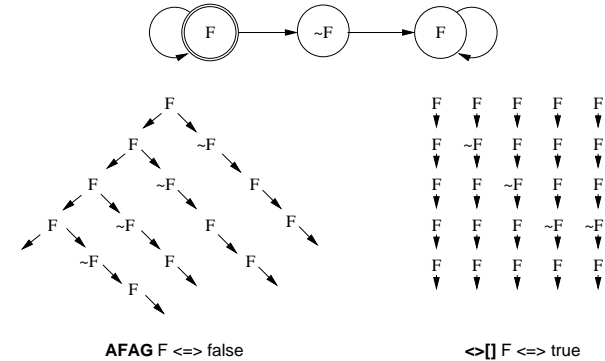
<http://www.risc.uni-linz.ac.at>

19/39

# Branching versus Linear Time Logic



Proof that **AFAG F** (CTL) is different from  $\diamond\Box F$  (LTL).



**AFAG F**  $\Leftrightarrow$  false  
 In every run, there is a moment when it is guaranteed that from now on F holds forever.

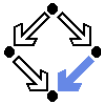
$\diamond\Box F$   $\Leftrightarrow$  true  
 In every run, there is a moment from which on F holds forever.

Wolfgang Schreiner

<http://www.risc.uni-linz.ac.at>

18/39

# Linear Time Logic



Why using linear time logic (LTL) for system specifications?

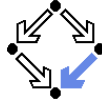
- LTL has many advantages:
  - LTL formulas are easier to understand.
    - Reasoning about computation paths, not computation trees.
    - No explicit path quantifiers used.
  - LTL can express most interesting system properties.
    - Invariance, guarantee, response, ... (see later).
  - LTL can express fairness constraints (see later).
    - CTL cannot do this.
    - But CTL can express that a state is reachable (which LTL cannot).
- LTL has also some disadvantages:
  - LTL is strictly less expressive than other specification languages.
    - CTL\* or  $\mu$ -calculus.
  - Asymptotic complexity of model checking is higher.
    - LTL: exponential in size of formula; CTL: linear in size of formula.
    - In practice the number of states dominates the checking time.

Wolfgang Schreiner

<http://www.risc.uni-linz.ac.at>

20/39

## Frequently Used LTL Patterns

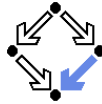


In practice, most temporal formulas are instances of particular patterns.

Pattern	Pronounced	Name
$\Box F$	always $F$	invariance
$\Diamond F$	eventually $F$	guarantee
$\Box \Diamond F$	$F$ holds infinitely often	recurrence
$\Diamond \Box F$	eventually $F$ holds permanently	stability
$\Box(F \Rightarrow \Diamond G)$	always, if $F$ holds, then eventually $G$ holds	response
$\Box(F \Rightarrow (G \mathbf{U} H))$	always, if $F$ holds, then $G$ holds until $H$ holds	precedence

Typically, there are at most two levels of nesting of temporal operators.

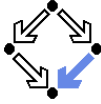
## Temporal Rules



Temporal operators obey a number of fairly intuitive rules.

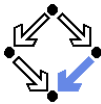
- **Extraction laws:**
  - $\Box F \Leftrightarrow F \wedge \Box F$ .
  - $\Diamond F \Leftrightarrow F \vee \Diamond F$ .
  - $F \mathbf{U} G \Leftrightarrow G \vee (F \wedge \Diamond(F \mathbf{U} G))$ .
- **Negation laws:**
  - $\neg \Box F \Leftrightarrow \Diamond \neg F$ .
  - $\neg \Diamond F \Leftrightarrow \Box \neg F$ .
  - $\neg(F \mathbf{U} G) \Leftrightarrow (\neg G) \mathbf{U} (\neg F \wedge \neg G)$ .
- **Distributivity laws:**
  - $\Box(F \wedge G) \Leftrightarrow (\Box F) \wedge (\Box G)$ .
  - $\Diamond(F \vee G) \Leftrightarrow (\Diamond F) \vee (\Diamond G)$ .
  - $(F \wedge G) \mathbf{U} H \Leftrightarrow (F \mathbf{U} H) \wedge (G \mathbf{U} H)$ .
  - $F \mathbf{U} (G \vee H) \Leftrightarrow (F \mathbf{U} G) \vee (F \mathbf{U} H)$ .
  - $\Box \Diamond(F \vee G) \Leftrightarrow (\Box \Diamond F) \vee (\Box \Diamond G)$ .
  - $\Diamond \Box(F \wedge G) \Leftrightarrow (\Diamond \Box F) \wedge (\Diamond \Box G)$ .

## Examples



- **Mutual exclusion:**  $\Box \neg(pc_1 = C \wedge pc_2 = C)$ .
  - Alternatively:  $\neg \Diamond(pc_1 = C \wedge pc_2 = C)$ .
  - Never both components are simultaneously in the critical region.
- **No starvation:**  $\forall i : \Box(pc_i = W \Rightarrow \Diamond pc_i = R)$ .
  - Always, if component  $i$  waits for a response, it eventually receives it.
- **No deadlock:**  $\Box \neg \forall i : pc_i = W$ .
  - Never all components are simultaneously in a wait state  $W$ .
- **Precedence:**  $\forall i : \Box(pc_i \neq C \Rightarrow (pc_i \neq C \mathbf{U} lock = i))$ .
  - Always, if component  $i$  is out of the critical region, it stays out until it receives the shared lock variable (which it eventually does).
- **Partial correctness:**  $\Box(pc = L \Rightarrow C)$ .
  - Always if the program reaches line  $L$ , the condition  $C$  holds.
- **Termination:**  $\forall i : \Diamond(pc_i = T)$ .
  - Every component eventually terminates.

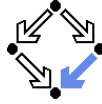
## Classes of System Properties



There exists two important classes of system properties.

- **Safety Properties:**
  - A safety property is a property such that, if it is violated by a run, it is already violated by some **finite prefix** of the run.
    - This finite prefix cannot be extended in any way to a complete run satisfying the property.
  - Example:  $\Box F$ .
    - The violating run  $F \rightarrow F \rightarrow \neg F \rightarrow \dots$  has the prefix  $F \rightarrow F \rightarrow \neg F$  that cannot be extended in any way to a run satisfying  $\Box F$ .
- **Liveness Properties:**
  - A liveness property is a property such that every finite prefix can be extended to a complete run satisfying this property.
    - Only a **complete run itself** can violate that property.
  - Example:  $\Diamond F$ .
    - Any finite prefix  $p$  can be extended to a run  $p \rightarrow F \rightarrow \dots$  which satisfies  $\Diamond F$ .

# System Properties

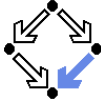


Not every system property is itself a safety property or a liveness property.

- **Example:**  $P := \Box A \wedge \Diamond B$ 
  - Conjunction of a safety property and a liveness property.
- Take the run  $[A, \neg B] \rightarrow [A, \neg B] \rightarrow [A, \neg B] \rightarrow \dots$  violating  $P$ .
  - Any prefix  $[A, \neg B] \rightarrow \dots \rightarrow [A, \neg B]$  of this run can be extended to a run  $[A, \neg B] \rightarrow \dots \rightarrow [A, \neg B] \rightarrow [A, B] \rightarrow [A, B] \rightarrow \dots$  satisfying  $P$ .
  - Thus  $P$  is **not a safety property**.
- Take the finite prefix  $[\neg A, B]$ .
  - This prefix cannot be extended in any way to a run satisfying  $P$ .
  - Thus  $P$  is **not a liveness property**.

So is the distinction “safety” versus “liveness” really useful?.

# System Properties



The real importance of the distinction is stated by the following theorem.

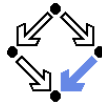
- **Theorem:**

Every system property  $P$  is a conjunction  $S \wedge L$  of some safety property  $S$  and some liveness property  $L$ .

  - If  $L$  is “true”, then  $P$  itself is a safety property.
  - If  $S$  is “true”, then  $P$  itself is a liveness property.
- **Consequence:**
  - Assume we can decompose  $P$  into appropriate  $S$  and  $L$ .
  - For proving  $M \models P$ , it then suffices to perform two proofs:
    - A safety proof:  $M \models S$ .
    - A liveness proof:  $M \models L$ .
  - Different strategies for proving safety and liveness properties.

For verification, it is important to decompose a system property in its “safety part” and its “liveness part”.

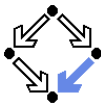
# Proving Invariance



We only consider a special case of a safety property.

- **Prove  $M \models \Box F$ .**
  - $F$  is a state formula (a formula without temporal operator).
  - Prove that  $F$  is an **invariant** of system  $M$ .
- $M = \langle I, R \rangle$ .
  - $I(s) := \dots$
  - $R(s, s') := R_0(s, s') \vee R_1(s, s') \vee \dots \vee R_{n-1}(s, s')$ .
- **Induction Proof.**
  - $\forall s : I(s) \Rightarrow F(s)$ .
    - Proof that  $F$  holds in every initial state.
  - $\forall s, s' : F(s) \wedge R(s, s') \Rightarrow F(s')$ .
    - Proof that each transition preserves  $F$ .
    - Reduces to a number of subproofs:
      - $F(s) \wedge R_0(s, s') \Rightarrow F(s')$
      - $\dots$
      - $F(s) \wedge R_{n-1}(s, s') \Rightarrow F(s')$

# Example



```

var x := 0
loop
  p0 : wait x = 0
  p1 : x := x + 1
||
loop
  q0 : wait x = 1
  q1 : x := x - 1
    
```

$$State = \{p_0, p_1\} \times \{q_0, q_1\} \times \mathbb{Z}.$$

$$I(p, q, x) := \Box (p = p_0 \wedge q = q_0 \wedge x = 0).$$

$$R(\langle p, q, x \rangle, \langle p', q', x' \rangle) := P_0(\dots) \vee P_1(\dots) \vee Q_0(\dots) \vee Q_1(\dots).$$

$$P_0(\langle p, q, x \rangle, \langle p', q', x' \rangle) := p = p_0 \wedge x = 0 \wedge p' = p_1 \wedge q' = q \wedge x' = x.$$

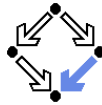
$$P_1(\langle p, q, x \rangle, \langle p', q', x' \rangle) := p = p_1 \wedge p' = p_0 \wedge q' = q \wedge x' = x + 1.$$

$$Q_0(\langle p, q, x \rangle, \langle p', q', x' \rangle) := q = q_0 \wedge x = 1 \wedge p' = p \wedge q' = q_1 \wedge x' = x.$$

$$Q_1(\langle p, q, x \rangle, \langle p', q', x' \rangle) := q = q_1 \wedge p' = p \wedge q' = q_0 \wedge x' = x - 1.$$

Prove  $\langle I, R \rangle \models \Box (x = 0 \vee x = 1)$ .

## Inductive System Properties

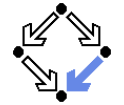


The induction strategy may not work for proving  $\Box F$

- **Problem:**  $F$  is **not inductive**.
  - $F$  is too weak to prove the induction step.
    - $F(s) \wedge R(s, s') \Rightarrow F(s')$ .
- **Solution:** find **stronger** invariant  $I$ .
  - If  $I \Rightarrow F$ , then  $(\Box I) \Rightarrow (\Box F)$ .
  - It thus suffices to prove  $\Box I$ .
- **Rationale:**  $I$  may be **inductive**.
  - If yes,  $I$  is strong enough to prove the induction step.
    - $I(s) \wedge R(s, s') \Rightarrow I(s')$ .
  - If not, find a stronger invariant  $I'$  and try again.
- Invariant  $I$  represents additional knowledge for every proof.
  - Rather than proving  $\Box P$ , prove  $\Box(I \Rightarrow P)$ .

The behavior of a system is captured by its strongest invariant.

## Example



- Prove  $\langle I, R \rangle \models \Box(x = 0 \vee x = 1)$ .
  - Proof attempt fails.
- Prove  $\langle I, R \rangle \models \Box G$ .
  - $G : \Leftrightarrow$ 

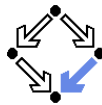
$$(x = 0 \vee x = 1) \wedge$$

$$(p = p_1 \Rightarrow x = 0) \wedge$$

$$(q = q_1 \Rightarrow x = 1).$$
  - Proof works.
  - $G \Rightarrow (x = 0 \vee x = 1)$  obvious.

See the proof presented in class.

## Proving Liveness



```

var x := 0, y := 0
loop
  x := x + 1
||
loop
  y := y + 1
    
```

$State = \mathbb{N} \times \mathbb{N}; Label = \{p, q\}$ .

$I(x, y) : \Leftrightarrow x = 0 \wedge y = 0$ .

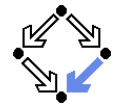
$R(I, \langle x, y \rangle, \langle x', y' \rangle) : \Leftrightarrow$

$(I = p \wedge x' = x + 1 \wedge y' = y) \vee (I = q \wedge x' = x \wedge y' = y + 1)$ .

- Prove  $\langle I, R \rangle \not\models \Diamond x = 1$ .
  - $[x = 0, y = 0] \rightarrow [x = 0, y = 1] \rightarrow [x = 0, y = 2] \rightarrow \dots$
  - This run violates (as the only one)  $\Diamond x = 1$ .
  - Thus the system as a whole does not satisfy  $\Diamond x = 1$ .

For proving liveness properties, “unfair” runs have to be ruled out.

## Enabling Condition



When is a particular transition enabled for execution?

- $Enabled_R(I, s) : \Leftrightarrow \exists t : R(I, s, t)$ .
  - Labeled transition relation  $R$ , label  $I$ , state  $s$ .
  - Read: “Transition (with label)  $I$  is enabled in state  $s$  (w.r.t.  $R$ )”.
- Example (previous slide):
  - $Enabled_R(p, \langle x, y \rangle)$ 

$$\Leftrightarrow \exists x', y' : R(p, \langle x, y \rangle, \langle x', y' \rangle)$$

$$\Leftrightarrow \exists x', y' :$$

$$(p = p \wedge x' = x + 1 \wedge y' = y) \vee$$

$$(p = q \wedge x' = x \wedge y' = y + 1)$$

$$\Leftrightarrow (\exists x', y' : p = p \wedge x' = x + 1 \wedge y' = y) \vee$$

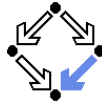
$$(\exists x', y' : p = q \wedge x' = x \wedge y' = y + 1)$$

$$\Leftrightarrow \text{true} \vee \text{false}$$

$$\Leftrightarrow \text{true}.$$
  - Transition  $p$  is always enabled.



## Weak Fairness

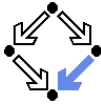


### Weak Fairness

- A run  $s_0 \xrightarrow{l_0} s_1 \xrightarrow{l_1} s_2 \xrightarrow{l_2} \dots$  is **weakly fair** to a transition  $l$ , if
  - if transition  $l$  is eventually **permanently** enabled in the run,
  - then transition  $l$  is executed infinitely often in the run.
 
$$(\exists i : \forall j \geq i : \text{Enabled}_R(l, s_j)) \Rightarrow (\forall i : \exists j \geq i : l_j = l).$$
  - The run in the previous example was not weakly fair to transition  $p$ .
- LTL formulas may **explicitly specify** weak fairness constraints.
  - Let  $E_l$  denote the enabling condition of transition  $l$ .
  - Let  $X_l$  denote the predicate “transition  $l$  is executed”.
  - Define  $WF_l := (\diamond \square E_l) \Rightarrow (\square \diamond X_l)$ .
    - If  $l$  is eventually enabled forever, it is executed infinitely often.
  - Prove  $\langle l, S \rangle \models (WF_l \Rightarrow P)$ .
    - Property  $P$  is only proved for runs that are weakly fair to  $l$ .

Alternatively, a model may also have weak fairness “built in”.

## Proving a Guarantee

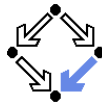


We only consider a special case of a liveness property.

- **Prove**  $\langle l, R \rangle \models \diamond F$ .
  - Proof that  $F$  is a **guarantee** of the system.
  - $F$  is a state formula (a formula without a temporal operator).
- **Decomposition:** sequence of properties  $F_0, F_1, \dots, F_n = F$ .
  - Prove  $\langle l, R \rangle \models \diamond F_0$ .
  - Prove  $\langle l, R \rangle \models \square (F_0 \Rightarrow \diamond F_1)$ .
  - Prove  $\langle l, R \rangle \models \square (F_1 \Rightarrow \diamond F_2)$ .
  - ...
  - Prove  $\langle l, R \rangle \models \square (F_{n-1} \Rightarrow \diamond F)$ .

Typically, guarantee proofs have to be decomposed into multiple proofs.

## Proving a Guarantee



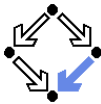
- **Core proof:**  $\langle l, R \rangle \models \diamond F$ .
  - Find **lucky transition**  $l$  with enabling condition  $E_l$ .
    - The execution of  $l$  makes  $F$  true.
    - As long as  $F$  is not true,  $l$  is enabled.
    - By weak fairness, either  $F$  becomes true or  $l$  is eventually executed.
    - Until  $l$  is executed, additional property  $H$  holds.
 
$$\neg F(s) \wedge l(s) \Rightarrow H(s) \wedge E_l(s).$$

$$\neg F(s) \wedge H(s) \wedge E_l(s) \wedge \neg R(l, s, s') \Rightarrow H(s') \wedge E_l(s').$$

$$\neg F(s) \wedge H(s) \wedge R(l, s, s') \Rightarrow F(s').$$
- **Core proofs:**  $\langle l, R \rangle \models \square (G \Rightarrow \diamond F)$ .
  - Find **lucky transition**  $l$  with enabling condition  $E_l$ .
    - Prove:  $\neg F(s) \wedge G(s) \Rightarrow H(s) \wedge E_l(s)$ .
    - Prove:  $\neg F(s) \wedge H(s) \wedge E_l(s) \wedge \neg R(l, s, s') \Rightarrow H(s') \wedge E_l(s')$ .
    - Prove:  $\neg G(s) \wedge H(s) \wedge R(l, s, s') \Rightarrow F(s')$ .

Sometimes augmented by proofs using well-founded orderings.

## Example



$State = \mathbb{N} \times \mathbb{N}; Label = \{p, q\}$ .

$l(x, y) := x = 0 \wedge y = 0$ .

$R(l, \langle x, y \rangle, \langle x', y' \rangle) :=$

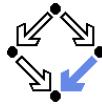
$(l = p \wedge x' = x + 1 \wedge y' = y) \vee (l = q \wedge x' = x \wedge y' = y + 1)$ .

- **Prove**  $\langle l, R \rangle \models \diamond x = 1$ .
  - Lucky transition  $p$ , additional property  $H := x = 0$ .
 
$$x \neq 1 \wedge (x = 0 \wedge y = 0) \Rightarrow x = 0 \wedge \text{true}.$$

$$x \neq 1 \wedge x = 0 \wedge \text{true} \wedge (x' = x \wedge y' = y + 1) \Rightarrow x' = 0 \wedge \text{true}.$$

$$x \neq 1 \wedge x = 0 \wedge (x' = x + 1 \wedge y' = y) \Rightarrow x' = 1.$$

## Strong Fairness

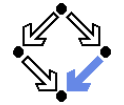


### Strong Fairness

- A run  $s_0 \xrightarrow{l_0} s_1 \xrightarrow{l_1} s_2 \xrightarrow{l_2} \dots$  is **strongly fair** to a transition  $l$ , if
  - if  $l$  is **infinitely often** enabled in the run,
  - then  $l$  is also infinitely often executed the run.  
 $(\forall i : \exists j \geq i : Enabled_R(l, s_j)) \Rightarrow (\forall i : \exists j \geq i : l_j = l)$ .
  - If  $r$  is strongly fair to  $l$ , it is also weakly fair to  $l$  (but not vice versa).
- LTL formulas may **explicitly specify** strong fairness constraints.
  - Let  $E_l$  denote the enabling condition of transition  $l$ .
  - Let  $X_l$  denote the predicate “transition  $l$  is executed”.
  - Define  $SF_l : \Leftrightarrow (\Box \Diamond E_l) \Rightarrow (\Box \Diamond X_l)$ .  
If  $l$  is enabled infinitely often, it is executed infinitely often.
  - Prove  $\langle l, S \rangle \models (SF_l \Rightarrow P)$ .  
Property  $P$  is only proved for runs that are strongly fair to  $l$ .

A much stronger requirement to the fairness of a system.

## Example

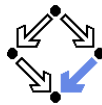


```
var x=0
loop
  a : x := -x
  b : choose x := 0 [] x := 1
```

$State := \{a, b\} \times \mathbb{Z}; Label = \{A, B_0, B_1\}.$   
 $I(p, x) : \Leftrightarrow p = a \wedge x = 0.$   
 $R(l, \langle p, x \rangle, \langle p', x' \rangle) : \Leftrightarrow$   
 $(l = A \wedge (p = a \wedge p' = b \wedge x' = -x)) \vee$   
 $(l = B_0 \wedge (p = b \wedge p' = a \wedge x' = 0)) \vee$   
 $(l = B_1 \wedge (p = b \wedge p' = a \wedge x' = 1)).$

- **Prove:**  $\langle l, R \rangle \models \Diamond x = 1.$ 
  - Take violating run  $[a, 0] \xrightarrow{A} [b, 0] \xrightarrow{B_0} [a, 0] \xrightarrow{A} [b, 0] \xrightarrow{B_0} [a, 0] \xrightarrow{A} \dots$
  - $Enabled_{B_1}(p, x) : \Leftrightarrow p = b.$
  - Run is weakly fair **but not strongly fair** to  $B_1$ .

## Weak versus Strong Fairness



In which situations is which notion of fairness appropriate?

- Multiple processes compete for resource that is permanently available.
  - CPU time, communication bandwidth, etc.
  - Weak fairness suffices.
- Multiple processes compete for resource that becomes temporarily unavailable by being in use.
  - Critical region protected by lock variable (mutex/semaphore).
  - Strong fairness is required.
- Non-deterministic choices are repeatedly made in program.
  - Simultaneous listing on multiple communication channels.
  - Strong fairness is required.

Many other notions of fairness exist.