The Language of Logic

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The Language of Logic

Two kinds of syntactic phrases.

- Term T denoting an object.
  - Variable x
  - Object constant c
  - Function application $f(T_1, \ldots, T_n)$
    - n-ary function constant f (may be infix)
- Formula F denoting a truth value.
  - Atomic formula $p(T_1, \ldots, T_n)$ (may be infix)
    - n-ary predicate constant p.
  - Negation $\neg F$ ("not F")
  - Conjunction $F_1 \wedge F_2$ ("F_1 and F_2")
  - Disjunction $F_1 \vee F_2$ ("F_1 or F_2")
  - Implication $F_1 \Rightarrow F_2$ ("F_1 implies F_2")
  - Equivalence $F_1 \Leftrightarrow F_2$ ("F_1 is equivalent to F_2")
  - Universal quantification $\forall x : F$ ("for all x, F")
  - Existential quantification $\exists x : F$ ("for some x, F")

Syntactic Shortcuts

- $\forall x_1, \ldots, x_n : F$
  - $\forall x_1 : \ldots : \forall x_n : F$
- $\exists x_1, \ldots, x_n : F$
  - $\exists x_1 : \ldots : \exists x_n : F$
- $\forall x \in S : F$
  - $\forall x : x \in S \Rightarrow F$
- $\exists x \in S : F$
  - $\exists x : x \in S \wedge F$
- **let** $x = T$ **in** F
  - $F[T/x]$

Help to make formulas more readable.

Example

Terms and formulas may appear in various syntactic forms.

- Terms:

  $\exp(x)$
  $a \cdot b + 1$
  $a[i] \cdot b$
  $\sqrt{\frac{x^2+2x+1}{(y+1)^2}}$
- Formulas:

  $a^2 + b^2 = c^2$
  $n \mid 2n$
  $\forall x \in \mathbb{N} : x \geq 0$
  $\forall x \in \mathbb{N} : 2|x \vee 2|(x+1)$
  $\forall x \in \mathbb{N}, y \in \mathbb{N} : x < y \Rightarrow$
  $\quad \exists z \in \mathbb{N} : x + z = y$

Terms and formulas may be nested arbitrarily deeply.

The Meaning of Formulas

- Atomic formula $p(T_1, \ldots, T_n)$
  - True if the predicate denoted by p holds for the values of $T_1, \ldots, T_n$.
- Negation $\neg F$
  - True if and only if F is false.
- Conjunction $F_1 \wedge F_2$ ("F_1 and F_2")
  - True if and only if F_1 and F_2 are both true.
- Disjunction $F_1 \vee F_2$ ("F_1 or F_2")
  - True if and only if at least one of F_1 or F_2 is true.
- Implication $F_1 \Rightarrow F_2$ ("F_1 implies F_2")
  - False if and only if F_1 is true and F_2 is false.
- Equivalence $F_1 \Leftrightarrow F_2$ ("F_1 is equivalent to F_2")
  - True if and only if F_1 and F_2 are both true or both false.
- Universal quantification $\forall x : F$ ("for all x, F")
  - True if and only if F is true for every possible value assignment of x.
- Existential quantification $\exists x : F$ ("for some x, F")
  - True if and only if F is true for at least one value assignment of x.

Example

We assume the domain of natural numbers and the "classical" interpretation of constants 1, 2, $+$, $=$, $<$.

- $1+1=2$
  - True.
- $1+1=2 \vee 2+2=2$
  - True.
- $1+1=2 \wedge 2+2=2$
  - False.
- $1+1=2 \Rightarrow 2=1+1$
  - True.
- $1+1=1 \Rightarrow 2+2=2$
  - True.
- $1+1=2 \Rightarrow 2+2=2$
  - False.
- $1+1=1 \Leftrightarrow 2+2=2$
  - True.

Example

- $x+1=1+x$
  - True, for every assignment of a number a to variable x.
- $\forall x : x+1=1+x$
  - True (because for every assignment a to x, $x+1=1+x$ is true).
- $x+1=2$
  - If x is assigned "one", the formula is true.
  - If x is assigned "two", the formula is false.
- $\exists x : x+1=2$
  - True (because $x+1=2$ is true for assignment "one" to x).
- $\forall x : x+1=2$
  - False (because $x+1=2$ is false for assignment "two" to x).
- $\forall x : \exists y : x < y$
  - True (because for every assignment a to x, there exists the assignment $a+1$ to y which makes $x < y$ true).
- $\exists y : \forall x : x < y$
  - False (because for every assignment a to y, there is the assignment $a+1$ to x which makes $x < y$ false).

The Usage of Formulas

Precise formulation of statements describing object relationships.

- Statement:

  If x and y are natural numbers and y is not zero, then q is the truncated quotient of x divided by y.
- Formula:

  $$x \in \mathbb{N} \wedge y \in \mathbb{N} \wedge y \neq 0 \Rightarrow$$
  $$q \in \mathbb{N} \wedge \exists r \in \mathbb{N} : r < y \wedge x = y \cdot q + r$$
- Problem specification:

  Given natural numbers x and y such that y is not zero, compute the truncated quotient q of x divided by y.

  - Inputs: x, y
  - Input condition: $x \in \mathbb{N} \wedge y \in \mathbb{N} \wedge y \neq 0$
  - Output: q
  - Output condition: $q \in \mathbb{N} \wedge \exists r \in \mathbb{N} : r < y \wedge x = y \cdot q + r$

Problem Specifications

- Specification of a computation problem:
  - Input: variables $X_1 \in S_1, \ldots, X_n \in S_n$
  - Input condition: formula I in which only $X_1, \ldots, X_n$ may be free.
  - Output: variables $Y_1 \in T_1, \ldots, Y_m \in T_n$
  - Output condition: formula O in which only $X_1, \ldots, X_n, Y_1, \ldots, Y_m$ may be free.

    A variable is free in a formula, if it occurs in the formula outside the scope of a quantifier (such as $\forall$ or $\exists$).
- Implementation of the specification:
  - A function (program) $f : S_1 \times \ldots \times S_n \rightarrow T_1 \times \ldots \times T_m$ such that

    $$\forall X_1 \in S_1, \ldots, X_n \in S_n : I(X_1, \ldots, X_n) \Rightarrow$$
    $$\textbf{let } (Y_1, \ldots, Y_m) = f(X_1, \ldots, X_n) \textbf{ in}$$
    $$O(X_1, \ldots, X_n, Y_1, \ldots, Y_m)$$
  - For all arguments that satisfy the input condition, f must compute results that satisfy the output condition.

Basis of all specification formalisms.

Example

Given an array a, a position p, and a length l, compute the array b derived from a by removing $a[p], \ldots, a[p+l]$.

- Input: $a \in \mathrm{Arr}$, $p \in \mathbb{N}$, $l \in \mathbb{N}$
- Input condition:

  $$p + l \leq \mathrm{length}(a)$$
- Output: $b \in \mathrm{Arr}$
- Output condition:

  $$\textbf{let } n = \mathrm{length}(a) \textbf{ in}$$
  $$\mathrm{length}(b) = n - l \wedge$$
  $$(\forall i \in \mathbb{N} : i < p \Rightarrow b[i] = a[i]) \wedge$$
  $$(\forall i \in \mathbb{N} : p \leq i < n - l \Rightarrow b[i] = a[i + l])$$

Proving Formulas

- Proof: a structured argument that a formula is true.
  - Can be depicted as a tree whose nodes represent proof situations.
  - Each proof situation consists of knowledge (formulas assumed to be true) and a goal (a formula to be proved relative to that knowledge).
  - The root goal is the overall formula to be proved.
- Proof rules: describe how a proof situation can be reduced to zero or more "subsituations".
  - Zero subsituations: the current goal has been proved.
  - One or more subsituations: the current goal has been proved, if all subgoals have been proved.
  - Top-down rules: focus on goal formula.
    - Goal formula is decomposed into simpler formulas.
  - Bottom-up rules: focus on some formula in knowledge.
    - Additional formulas are added to the knowledge.

In each proof situation, we aim at showing that the goal is "apparently" true with respect to the given knowledge.

Conjunction

Formula $F_1 \wedge F_2$.

- Formula as goal.
  - Create two subsituations with goals F_1 and F_2.

    We have to show $F_1 \wedge F_2$.
    - *We show F_1: . . . (proof continues with goal F_1)*
    - *We show F_2: . . . (proof continues with goal F_2)*
- Formula in knowledge.
  - Create one subsituation with F_1 and F_2 in knowledge.

    We know $F_1 \wedge F_2$. We thus know F_1 and also F_2. (proof continues with current goal and additional knowledge F_1 and F_2)

Disjunction

Formula $F_1 \vee F_2$.

- Formula as goal.
  - Create one subsituation where F_2 is proved under the assumption that F_1 does not hold (or vice versa):

    We have to show $F_1 \vee F_2$. We assume $\neg F_1$ and show F_2. (proof continues with goal F_2 and additional knowledge $\neg F_1$)
- Formula in knowledge.
  - Create two subsituations, one with F_1 and one with F_2 in knowledge.

    We know $F_1 \vee F_2$. We thus proceed by case distinction:
    - *Case F_1: ... (proof continues with current goal and additional knowledge F_1).*
    - *Case F_2: ... (proof continues with current goal and additional knowledge F_2).*

Implication

Formula $F_1 \Rightarrow F_2$.

- Formula as goal.
  - Create one subsituation where F_2 is proved under the assumption that F_1 holds:

    We have to show $F_1 \Rightarrow F_2$. We assume F_1 and show F_2. (proof continues with goal F_2 and additional knowledge F_1)
- Formula in knowledge.
  - Create two subsituations, one with goal F_1 and one with knowledge F_2.

    We know $F_1 \Rightarrow F_2$.
    - *We show F_1: ... (proof continues with goal F_1)*
    - *We know F_2: ... (proof continues with current goal and additional knowledge F_2).*

Equivalence

Formula $F_1 \Leftrightarrow F_2$.

- Formula as goal.
  - Create two subsituations with implications in both directions as goals:

    We have to show $F_1 \Leftrightarrow F_2$.
    - *We show $F_1 \Rightarrow F_2$: ... (proof continues with goal $F_1 \Rightarrow F_2$)*
    - *We show $F_2 \Rightarrow F_1$: ... (proof continues with goal $F_2 \Rightarrow F_1$)*
- Formula in knowledge.
  - Create two subsituations, one with goal F_1 and one with knowledge F_2.

    We know $F_1 \Leftrightarrow F_2$.
    - *We show F_1 ($\neg F_1$): ... (proof continues with goal F_1)*
    - *We know F_2 ($\neg F_2$): ... (proof continues with current goal and additional knowledge F_2)*

Universal Quantification

Formula $\forall x : F$

- Formula as goal.
  - Introduce new (arbitrarily named) constant x_0 and create one subsituation with goal $F[x_0/x]$.

    We have to show $\forall x : F$. Take arbitrary x_0. We show $F[x_0/x]$. (proof continues with goal $F[x_0/x]$)
- Formula in knowledge.
  - Choose term T to create one subsituation with formula $F[T/x]$ added to the knowledge.

    We know $\forall x : F$ and thus also $F[T/x]$. (proof continues with current goal and additional knowledge $F[T/x]$)

Existential Quantification

Formula $\exists x : F$

- Formula as goal.
  - Choose term T to create one subsituation with goal $F[T/x]$.

    We have to show $\exists x : F$. It suffices to show $F[T/x]$.
    (proof continues with goal $F[T/x]$)
- Formula in knowledge
  - Introduce new (arbitrarily named constant) x_0 and create one subsituation with additional knowledge $F[x_0/x]$.

    We know $\exists x : F$. Let x_0 be such that $F[x_0/x]$.
    (proof continues with current goal and additional knowledge $F[x_0/x]$)

Formula Equivalences

Formulas in knowledge and goals may be replaced by equivalent formulas.

- $\neg\neg F_1 \leftrightsquigarrow F_1$
- $\neg(F_1 \wedge F_2) \leftrightsquigarrow \neg F_1 \vee \neg F_2$
- $\neg(F_1 \vee F_2) \leftrightsquigarrow \neg F_1 \wedge \neg F_2$
- $\neg(F_1 \Rightarrow F_2) \leftrightsquigarrow F_1 \wedge \neg F_2$
- $\neg\forall x : F \leftrightsquigarrow \exists x : \neg F$
- $\neg\exists x : F \leftrightsquigarrow \forall x : \neg F$
- $F_1 \Rightarrow F_2 \leftrightsquigarrow \neg F_2 \Rightarrow \neg F_1$
- $F_1 \Rightarrow F_2 \leftrightsquigarrow \neg F_1 \vee F_2$
- $F_1 \Leftrightarrow F_2 \leftrightsquigarrow \neg F_1 \Leftrightarrow \neg F_2$
- ...

Transformation sometimes offers new view on a proof situation.

Indirect Proofs

Proof situation with goal formula G.

- Add $\neg G$ to the knowledge and show a contradiction.
  - Prove that "false" is true.
  - Prove that a formula F is true and also prove that it is false.
  - Prove that a formula F in the knowledge is false, i.e. that $\neg F$ is true.
    - Switches goal G and some knowledge F (negating both).

Sometimes simpler than a direct proof.

Example

We show

(a) $(\exists x : p(x)) \wedge (\forall x : p(x) \Rightarrow \exists y : q(x,y)) \Rightarrow (\exists x, y : q(x,y))$

We assume

(1) $(\exists x : p(x)) \wedge (\forall x : p(x) \Rightarrow \exists y : q(x,y))$

and show

(b) $\exists x, y : q(x,y)$

From (1), we know

(2) $\exists x : p(x)$
(3) $\forall x : p(x) \Rightarrow \exists y : q(x,y)$

From (2) we know for some x_0

(4) $p(x_0)$

...

Example (Contd)

...

From (3), we know

(5) $p(x_0) \Rightarrow \exists y : q(x_0, y)$

From (4) and (5), we know

(6) $\exists y : q(x_0, y)$

From (6), we know for some y_0

(7) $q(x_0, y_0)$

From (7), we know (b). QED.

Example

We show

(a) $(\exists x : \forall y : P(x, y)) \Rightarrow (\forall y : \exists x : P(x, y))$

We assume

(1) $\exists x : \forall y : P(x, y)$

and show

(b) $\forall y : \exists x : P(x, y)$

Take arbitrary y_0. We show

(c) $\exists x : P(x, y_0)$

From (1) we know for some x_0

(2) $\forall y : P(x_0, y)$

From (2) we know

(3) $P(x_0, y_0)$

From (3), we know (c). QED.

Example (Indirect Proof)

We show

(a) $(\exists x : \forall y : P(x, y)) \Rightarrow (\forall y : \exists x : P(x, y))$

We assume

(1) $\exists x : \forall y : P(x, y)$

and show

(b) $\forall y : \exists x : P(x, y)$

We assume

(2) $\neg \forall y : \exists x : P(x, y)$

and show a contradiction.

...

Example (Indirect Proof Contd)

...

From (2), we know

(3) $\exists y : \forall x : \neg P(x, y)$

Let y_0 be such that

(4) $\forall x : \neg P(x, y_0)$

From (1) we know for some x_0

(5) $\forall y : P(x_0, y)$

From (5) we know

(6) $P(x_0, y_0)$

From (4), we know

(7) $\neg P(x_0, y_0)$

From (6) and (7), we have a contradiction. QED.