Hoare Calculus and Predicate Transformers

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1. **The Hoare Calculus for Non-Loop Programs**
2. Predicate Transformers
3. Partial Correctness of Loop Programs
4. Total Correctness of Loop Programs
5. Abortion
6. Procedures

The Hoare Calculus

Calculus for reasoning about imperative programs.

- "Hoare triple": $\{P\}\ c\ \{Q\}$
  - Logical propositions P and Q, program command c.
  - The Hoare triple is itself a logical proposition.
  - The Hoare calculus gives rules for constructing true Hoare triples.
- Partial correctness interpretation of $\{P\}\ c\ \{Q\}$:

  "If c is executed in a state in which P holds, then it terminates in a state in which Q holds unless it aborts or runs forever."

  - Program does not produce wrong result.
  - But program also need not produce any result.
    - Abortion and non-termination are not ruled out.
- Total correctness interpretation of $\{P\}\ c\ \{Q\}$:

  "If c is executed in a state in which P holds, then it terminates in a state in which Q holds.

  - Program produces the correct result.

We will use the partial correctness interpretation for the moment.

General Rules

$$\frac{P \Rightarrow Q}{\{P\}\ \{Q\}} \qquad \frac{P \Rightarrow P' \quad \{P'\}\ c\ \{Q'\} \quad Q' \Rightarrow Q}{\{P\}\ c\ \{Q\}}$$

- Logical derivation: $\frac{A_1 \ A_2}{B}$
  - Forward: If we have shown A_1 and A_2, then we have also shown B.
  - Backward: To show B, it suffices to show A_1 and A_2.
- Interpretation of above sentences:
  - To show that, if P holds in a state, then Q holds in the same state (no command is executed), it suffices to show P implies Q.
    - Hoare triples are ultimately reduced to classical logic.
  - To show that, if P holds, then Q holds after executing c, it suffices to show this for a P' weaker than P and a Q' stronger than Q.
    - Precondition may be weakened, postcondition may be strengthened.

Special Commands

Commands modeling "emptiness" and abortion.

$$\{P\}\ \textbf{skip}\ \{P\} \qquad \{\text{true}\}\ \textbf{abort}\ \{\text{false}\}$$

- The **skip** command does not change the state; if P holds before its execution, then P thus holds afterwards as well.
- The **abort** command aborts execution and thus trivially satisfies partial correctness.
  - Axiom implies $\{P\}$ **abort** $\{Q\}$ for arbitrary P, Q.

Useful commands for reasoning and program transformations.

Scalar Assignments

$$\{Q[e/x]\}\ x := e\ \{Q\}$$

- Syntax
  - Variable x, expression e.
  - $Q[e/x] \ldots Q$ where every free occurrence of x is replaced by e.
- Interpretation
  - To make sure that Q holds for x after the assignment of e to x, it suffices to make sure that Q holds for e before the assignment.
- Partial correctness
  - Evaluation of e may abort.

$$\begin{array}{rcc} \{x+3<5\} & x := x+3 & \{x<5\} \\ \{x<2\} & x := x+3 & \{x<5\} \end{array}$$

Array Assignments

$$\{Q[a[i \mapsto e]/a]\}\ a[i] := e\ \{Q\}$$

- An array is modelled as a function $a : I \to V$
  - Index set I, value set V.
  - $a[i] = e$ ... a holds at index i the value e.
- Updated array $a[i \mapsto e]$
  - Array that is constructed from a by mapping index i to value e.
  - Axioms (for all $a : I \to V, i \in I, j \in I, e \in V$):

$$i = j \Rightarrow a[i \mapsto e][j] = e$$
$$i \neq j \Rightarrow a[i \mapsto e][j] = a[j]$$

$$\{a[i \mapsto x][1] > 0\} \quad a[i] := x \quad \{a[1] > 0\}$$
$$\{(i = 1 \Rightarrow x > 0) \wedge (i \neq 1 \Rightarrow a[1] > 0)\} \quad \text{a[i] := x} \quad \{a[1] > 0\}$$

Index violations and pointer semantics of arrays not yet considered.

Command Sequences

$$\frac{\{P\}\ c_1\ \{R_1\} \quad R_1 \Rightarrow R_2 \quad \{R_2\}\ c_2\ \{Q\}}{\{P\}\ c_1; c_2\ \{Q\}}$$

- Interpretation
  - To show that, if P holds before the execution of $c_1; c_2$, then Q holds afterwards, it suffices to show for some R_1 and R_2 with $R_1 \Rightarrow R_2$ that
    - if P holds before c_1, that R_1 holds afterwards, and that
    - if R_2 holds before c_2, then Q holds afterwards.
- Problem: find suitable R_1 and R_2
  - Easy in many cases (see later).

$$\frac{\{x+y-1>0\}\ y := y-1\ \{x+y>0\} \quad \{x+y>0\}\ x := x+y\ \{x>0\}}{\{x+y-1>0\}\ y := y-1; x := x+y\ \{x>0\}}$$

Conditionals

$$\frac{\{P \land b\}\ c_1\ \{Q\} \quad \{P \land \neg b\}\ c_2\ \{Q\}}{\{P\}\ \textbf{if}\ b\ \textbf{then}\ c_1\ \textbf{else}\ c_2\ \{Q\}}$$

$$\frac{\{P \land b\}\ c\ \{Q\} \quad (P \land \neg b) \Rightarrow Q}{\{P\}\ \textbf{if}\ b\ \textbf{then}\ c\ \{Q\}}$$

- Interpretation
  - To show that, if P holds before the execution of the conditional, then Q holds afterwards,
  - it suffices to show that the same is true for each conditional branch, under the additional assumption that this branch is executed.

$$\frac{\{x \neq 0 \land x \geq 0\}\ y := x\ \{y > 0\} \quad \{x \neq 0 \land x \not\geq 0\}\ y := -x\ \{y > 0\}}{\{x \neq 0\}\ \textbf{if}\ x \geq 0\ \textbf{then}\ y := x\ \textbf{else}\ y := -x\ \{y > 0\}}$$

Backward Reasoning

Implication of rule for command sequences and rule for assignments:

$$\frac{\{P\}\ c\ \{Q[e/x]\}}{\{P\}\ c; x := e\ \{Q\}}$$

- Interpretation
  - If the last command of a sequence is an assignment, we can remove the assignment from the proof obligation.
  - By multiple application, assignment sequences can be removed from the back to the front.

$\{P\}$	$\{P\}$	$\{P\}$	$\{P\}$	$P \Rightarrow x = 4$
x := x+1;	x := x+1;	x := x+1;	$\{x + 1 = 5\}$	
y := 2*x;	y := 2*x;	$\{x + 2x = 15\}$	$(\Leftrightarrow x = 4)$	
z := x+y	$\{x + y = 15\}$	$(\Leftrightarrow 3x = 15)$		
$\{z = 15\}$		$(\Leftrightarrow x = 5)$		

Weakest Preconditions

A calculus for "backward reasoning".

- Predicate transformer wp
  - Function "wp" that takes a command c and a postcondition Q and returns a precondition.
  - Read $\text{wp}(c, Q)$ as "the weakest precondition of c w.r.t. Q".
- $\text{wp}(c, Q)$ is a precondition for c that ensures Q as a postcondition.
  - Must satisfy $\{\text{wp}(c, Q)\}\ c\ \{Q\}$.
- $\text{wp}(c, Q)$ is the weakest such precondition.
  - Take any P such that $\{P\}\ c\ \{Q\}$.
  - Then $P \Rightarrow \text{wp}(c, Q)$.
- Consequence: $\{P\}\ c\ \{Q\}$ iff $(P \Rightarrow \text{wp}(c, Q))$
  - We want to prove $\{P\}\ c\ \{Q\}$.
  - We may prove $P \Rightarrow \text{wp}(c, Q)$ instead.

Verification is reduced to the calculation of weakest preconditions.

Weakest Preconditions

The weakest precondition of each program construct.

$$
\begin{aligned}
&\mathrm{wp}(\textbf{skip}, Q) \Leftrightarrow Q \\
&\mathrm{wp}(\textbf{abort}, Q) \Leftrightarrow \mathrm{true} \\
&\mathrm{wp}(x := e, Q) \Leftrightarrow Q[e/x] \\
&\mathrm{wp}(c_1; c_2, Q) \Leftrightarrow \mathrm{wp}(c_1, \mathrm{wp}(c_2, Q)) \\
&\mathrm{wp}(\textbf{if}\ b\ \textbf{then}\ c_1\ \textbf{else}\ c_2, Q) \Leftrightarrow (b \Rightarrow \mathrm{wp}(c_1, Q)) \wedge (\neg b \Rightarrow \mathrm{wp}(c_2, Q)) \\
&\mathrm{wp}(\textbf{if}\ b\ \textbf{then}\ c, Q) \Leftrightarrow (b \Rightarrow \mathrm{wp}(c, Q)) \wedge (\neg b \Rightarrow Q)
\end{aligned}
$$

Alternative formulation of a program calculus.

Forward Reasoning

Sometimes, we want to derive a postcondition from a given precondition.

$$\{P\}\ x := e\ \{\exists x_0 : P[x_0/x] \land x = e[x_0/x]\}$$

- Forward Reasoning
  - What is the maximum we know about the post-state of an assignment $x := e$, if the pre-state satisfies P?
  - We know that P holds for some value x_0 (the value of x in the pre-state) and that x equals $e[x_0/x]$.

$$\begin{gathered}\{x \geq 0 \land y = a\} \\ x := x + 1 \\ \{\exists x_0 : x_0 \geq 0 \land y = a \land x = x_0 + 1\} \\ (\Leftrightarrow (\exists x_0 : x_0 \geq 0 \land x = x_0 + 1) \land y = a) \\ (\Leftrightarrow x > 0 \land y = a)\end{gathered}$$

Strongest Postcondition

A calculus for forward reasoning.

- Predicate transformer sp
  - Function "sp" that takes a precondition P and a command c and returns a postcondition.
  - Read $\text{sp}(P, c)$ as "the strongest postcondition of c w.r.t. P".
- $\text{sp}(P, c)$ is a postcondition for c that is ensured by precondition P.
  - Must satisfy $\{P\}\ c\ \{\text{sp}(P, c)\}$.
- $\text{sp}(P, c)$ is the strongest such postcondition.
  - Take any P, Q such that $\{P\}\ c\ \{Q\}$.
  - Then $\text{sp}(P, c) \Rightarrow Q$.
- Consequence: $\{P\}\ c\ \{Q\}$ iff $(\text{sp}(P, c) \Rightarrow Q)$.
  - We want to prove $\{P\}\ c\ \{Q\}$.
  - We may prove $\text{sp}(P, c) \Rightarrow Q$ instead.

Verification is reduced to the calculation of strongest postconditions.

Strongest Postconditions

The strongest postcondition of each program construct.

$$\begin{aligned}
&\mathrm{sp}(P, \textbf{skip}) \Leftrightarrow P\\
&\mathrm{sp}(P, \textbf{abort}) \Leftrightarrow \mathrm{false}\\
&\mathrm{sp}(P, x := e) \Leftrightarrow \exists x_0 : P[x_0/x] \wedge x = e[x_0/x]\\
&\mathrm{sp}(P, c_1; c_2) \Leftrightarrow sp(sp(P, c_1), c_2)\\
&\mathrm{sp}(P, \textbf{if } b \textbf{ then } c_1 \textbf{ else } c_2) \Leftrightarrow \mathrm{sp}(P \wedge b, c_1) \vee \mathrm{sp}(P \wedge \neg b, c_2)\\
&\mathrm{sp}(P, \textbf{if } b \textbf{ then } c) \Leftrightarrow \mathrm{sp}(P \wedge b, c) \vee (P \wedge \neg b)
\end{aligned}$$

The use of predicate transformers is an alternative/supplement to the Hoare calculus; this view is due to Dijkstra.

The Hoare Calculus and Loops

$$\{\text{true}\}\ \textbf{loop}\ \{\text{false}\} \qquad \frac{P \Rightarrow I \quad \{I \wedge b\}\ c\ \{I\} \quad (I \wedge \neg b) \Rightarrow Q}{\{P\}\ \textbf{while}\ b\ \textbf{do}\ c\ \{Q\}}$$

- Interpretation:
  - The **loop** command does not terminate and thus trivially satisfies partial correctness.
    - Axiom implies $\{P\}$ **loop** $\{Q\}$ for arbitrary P, Q.
  - To show that, if before the execution of a **while**-loop the property P holds, after its termination the property Q holds, it suffices to show for some property I (the loop invariant) that
    - I holds before the loop is executed (i.e. that P implies I),
    - if I holds when the loop body is entered (i.e. if also b holds), that after the execution of the loop body I still holds,
    - when the loop terminates (i.e. if b does not hold), I implies Q.
- Problem: find appropriate loop invariant I.
  - Strongest relationship between all variables modified in loop body.

Example

$$I :\Leftrightarrow s = \sum_{j=1}^{i-1} j \wedge (n \geq 0 \Rightarrow 1 \leq i \leq n+1) \wedge (n < 0 \Rightarrow i = 1)$$

$$\frac{\begin{array}{c}(i = 1 \wedge s = 0) \Rightarrow I \\ \{I \wedge i \leq n\}\ s := s + i; i := i + 1\ \{I\} \\ (I \wedge i \not\leq n) \Rightarrow s = \sum_{j=1}^{n} j\end{array}}{\{i = 1 \wedge s = 0\}\ \textbf{while}\ i \leq n\ \textbf{do}\ (s := s + i; i := i + 1)\ \{s = \sum_{j=1}^{n} j\}}$$

The invariant captures the "essence" of a loop; only by giving its invariant, a true understanding of a loop is demonstrated.

Practical Aspects

We want to verify the following program:

$$\{P\}\ c_1; \textbf{while}\ b\ \textbf{do}\ c; c_2\ \{Q\}$$

- Assume c_1 and c_2 do not contain loop commands.
- It suffices to prove

$$\{\mathrm{sp}(P, c_1)\}\ \textbf{while}\ b\ \textbf{do}\ c\ \{\mathrm{wp}(c_2, Q)\}$$

Verification of loops is the core of most program verifications.

Weakest Liberal Preconditions for Loops

$$\text{wp}(\textbf{loop}, Q) \Leftrightarrow \text{true}$$
$$\text{wp}(\textbf{while } b \textbf{ do } c, Q) \Leftrightarrow \forall i \in \mathbb{N} : L_i(Q)$$

$$L_0(Q) :\Leftrightarrow \text{true}$$
$$L_{i+1}(Q) :\Leftrightarrow (\neg b \Rightarrow Q) \wedge (b \Rightarrow \text{wp}(c, L_i(Q)))$$

- Interpretation
  - Weakest precondition that ensures that loops stops in a state satisfying Q, unless it aborts or runs forever.
- Infinite sequence of predicates $L_i(Q)$:
  - Weakest precondition that ensures that after less than i iterations the state satisfies Q, unless the loop aborts or does not yet terminate.
- Alternative view: $L_i(Q) \Leftrightarrow \text{wp}(\text{if}_i, Q)$

$$\text{if}_0 := \textbf{loop}$$
$$\text{if}_{i+1} := \textbf{if } b \textbf{ then } (c; \text{if}_i)$$

Example

$$\mathrm{wp}(\textbf{while } i < n \textbf{ do } i := i+1, Q)$$

$$\begin{aligned}
L_0(Q) &\Leftrightarrow \text{true} \\
L_1(Q) &\Leftrightarrow (i \not< n \Rightarrow Q) \wedge (i < n \Rightarrow \mathrm{wp}(i := i+1, \text{true})) \\
&\Leftrightarrow (i \not< n \Rightarrow Q) \wedge (i < n \Rightarrow \text{true}) \\
&\Leftrightarrow (i \not< n \Rightarrow Q) \\
L_2(Q) &\Leftrightarrow (i \not< n \Rightarrow Q) \wedge (i < n \Rightarrow \mathrm{wp}(i := i+1, i \not< n \Rightarrow Q)) \\
&\Leftrightarrow (i \not< n \Rightarrow Q) \wedge \\
&\qquad (i < n \Rightarrow (i+1 \not< n \Rightarrow Q[i+1/i])) \\
L_3(Q) &\Leftrightarrow (i \not< n \Rightarrow Q) \wedge (i < n \Rightarrow \mathrm{wp}(i := i+1, \\
&\qquad (i \not< n \Rightarrow Q) \wedge (i < n \Rightarrow (i+1 \not< n \Rightarrow Q[i+1/i])))) \\
&\Leftrightarrow (i \not< n \Rightarrow Q) \wedge \\
&\qquad (i < n \Rightarrow ((i+1 \not< n \Rightarrow Q[i+1/i]) \wedge \\
&\qquad\qquad (i+1 < n \Rightarrow (i+2 \not< n \Rightarrow Q[i+2/i]))))
\end{aligned}$$

Weakest Liberal Preconditions for Loops

- Sequence $L_i(Q)$ is monotonically increasing in strength:
  - $\forall i \in \mathbb{N} : L_{i+1}(Q) \Rightarrow L_i(Q)$.
- The weakest precondition is the "lowest upper bound":
  - $\forall i \in \mathbb{N} : \text{wp}(\textbf{while } b \textbf{ do } c, Q) \Rightarrow L_i(Q)$.
  - $\forall P : (\forall i \in \mathbb{N} : P \Rightarrow L_i(Q)) \Rightarrow (P \Rightarrow \text{wp}(\textbf{while } b \textbf{ do } c, Q))$.
- We can only compute weaker approximation $L_i(Q)$.
  - $\text{wp}(\textbf{while } b \textbf{ do } c, Q) \Rightarrow L_i(Q)$.
- We want to prove $\{P\}$ **while** b **do** c $\{Q\}$.
  - This is equivalent to proving $P \Rightarrow \text{wp}(\textbf{while } b \textbf{ do } c, Q)$.
  - Thus $P \Rightarrow L_i(Q)$ must hold as well.
- If we can prove $\neg(P \Rightarrow L_i(Q))$, ...
  - $\{P\}$ **while** b **do** c $\{Q\}$ does not hold.
  - If we fail, we may try the easier proof $\neg(P \Rightarrow L_{i+1}(Q))$.

Falsification is possible by use of approximation L_i, but verification is not.

Total Correctness of Loops

Hoare rules for **loop** and **while** are replaced as follows:

$$\{\text{false}\}\ \textbf{loop}\ \{\text{false}\} \qquad \frac{P \Rightarrow I \quad I \wedge b \Rightarrow t > 0 \quad \{I \wedge b \wedge t = N\}\ c\ \{I \wedge t < N\} \quad (I \wedge \neg b) \Rightarrow Q}{\{P\}\ \textbf{while}\ b\ \textbf{do}\ c\ \{Q\}}$$

- New interpretation of $\{P\}\ c\ \{Q\}$.
  - If execution of c starts in a state where P holds, then execution terminates in a state where Q holds, unless it aborts.
  - Non-termination is ruled out, abortion not (yet).
  - The **loop** command thus does not satisfy total correctness.
- Termination term t.
  - Denotes a natural number before and after every loop iteration.
  - If $t = N$ before an iteration, then $t < N$ after the iteration.
  - Consequently, if term denotes zero, loop must terminate.

Instead of the natural numbers, any *well-founded ordering* may be used for the domain of t.

Example

$$I :\Leftrightarrow s = \sum_{j=1}^{i-1} j \wedge (n \geq 0 \Rightarrow 1 \leq i \leq n+1) \wedge (n < 0 \Rightarrow i = 1)$$

$$\frac{\begin{array}{c}(i = 1 \wedge s = 0) \Rightarrow I \quad I \wedge i \leq n \Rightarrow n - i + 1 > 0 \\ \{I \wedge i \leq 0 \wedge n - i + 1 = N\}\ s := s + i; i := i + 1\ \{I \wedge n - i + 1 < N\} \\ (I \wedge i \not\leq n) \Rightarrow s = \sum_{j=1}^{n} j\end{array}}{\{i = 1 \wedge s = 0\}\ \textbf{while}\ i \leq n\ \textbf{do}\ (s := s + i; i := i + 1)\ \{s = \sum_{j=1}^{n} j\}}$$

In practice, termination is easy to show (compared to partial correctness).

Weakest Preconditions for Loops

$\text{wp}(\textbf{loop}, Q) \Leftrightarrow \text{false}$
$\text{wp}(\textbf{while } b \textbf{ do } c, Q) \Leftrightarrow \exists i \in \mathbb{N} : L_i(Q)$

$L_0(Q) :\Leftrightarrow \text{false}$
$L_{i+1}(Q) :\Leftrightarrow (\neg b \Rightarrow Q) \wedge (b \Rightarrow \text{wp}(c, L_i(Q)))$

- New interpretation
  - Weakest precondition that ensures that the loop terminates in a state in which Q holds, unless it aborts.
- New interpretation of $L_i(Q)$
  - Weakest precondition that ensures that the loop terminates after less than i iterations in a state in which Q holds, unless it aborts.
- Preserves property: $\{P\}\ c\ \{Q\}$ iff $(P \Rightarrow \text{wp}(c, Q))$
  - Now for total correctness interpretation of Hoare calculus.
- Preserves alternative view: $L_i(Q) \Leftrightarrow \text{wp}(\text{if}_i, Q)$

  $\text{if}_0 := \textbf{loop}$
  $\text{if}_{i+1} := \textbf{if } b \textbf{ then } (c; \text{if}_i)$

Example

$$\text{wp}(\textbf{while } i < n \textbf{ do } i := i+1, Q)$$

$$
\begin{aligned}
L_0(Q) &:\Leftrightarrow \text{false} \\
L_1(Q) &:\Leftrightarrow (i \not< n \Rightarrow Q) \land (i < n \Rightarrow \ wp(i := i+1, L_0(Q))) \\
&\Leftrightarrow (i \not< n \Rightarrow Q) \land (i < n \Rightarrow \ \text{false}) \\
&\Leftrightarrow i \not< n \land Q \\
L_2(Q) &:\Leftrightarrow (i \not< n \Rightarrow Q) \land (i < n \Rightarrow \ wp(i := i+1, L_1(Q))) \\
&\Leftrightarrow (i \not< n \Rightarrow Q) \land \\
&\qquad i < n \Rightarrow (i+1 \not< n \land Q[i+1/i])) \\
L_3(Q) &:\Leftrightarrow (i \not< n \Rightarrow Q) \land (i < n \Rightarrow \ \text{wp}(i := i+1, L_2(Q))) \\
&\Leftrightarrow (i \not< n \Rightarrow Q) \land \\
&\qquad (i < n \Rightarrow \ ((i+1 \not< n \Rightarrow Q[i+1/i]) \land \\
&\qquad\qquad (i+1 < n \Rightarrow (i+2 \not< n \land Q[i+2/i]))))
\end{aligned}
$$

...

Weakest Preconditions for Loops

- Sequence $L_i(Q)$ is now monotonically decreasing in strength:
  - $\forall i \in \mathbb{N} : L_i(Q) \Rightarrow L_{i+1}(Q)$.
- The weakest precondition is the "greatest lower bound":
  - $\forall i \in \mathbb{N} : L_i(Q) \Rightarrow \text{wp}(\textbf{while } b \textbf{ do } c, Q)$.
  - $\forall P : (\forall i \in \mathbb{N} : L_i(Q) \Rightarrow P) \Rightarrow (\text{wp}(\textbf{while } b \textbf{ do } c, Q) \Rightarrow P)$.
- We can only compute a stronger approximation $L_i(Q)$.
  - $L_i(Q) \Rightarrow \text{wp}(\textbf{while } b \textbf{ do } c, Q)$.
- We want to prove $\{P\}\ c\ \{Q\}$.
  - It suffices to prove $P \Rightarrow \text{wp}(\textbf{while } b \textbf{ do } c, Q)$.
  - It thus also suffices to prove $P \Rightarrow L_i(Q)$.
  - If proof fails, we may try the easier proof $P \Rightarrow L_{i+1}(Q)$

However, verifications are typically not successful with finite approximation of weakest precondition.

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Abortion

New rules to prevent abortion.

$$\{\text{false}\}\ \textbf{abort}\ \{\text{true}\}$$
$$\{Q[e/x] \wedge D(e)\}\ x := e\ \{Q\}$$
$$\{Q[a[i \mapsto e]/a] \wedge D(e) \wedge 0 \leq i < \text{length}(a)\}\ a[i] := e\ \{Q\}$$

- New interpretation of $\{P\}\ c\ \{Q\}$.
  - If execution of c starts in a state, in which property P holds, then it does not abort and eventually terminates in a state in which Q holds.
- Sources of abortion.
  - Division by zero.
  - Index out of bounds exception.

$D(e)$ makes sure that every subexpression of e is well defined.

Definedness of Expressions

$D(0) :\Leftrightarrow \text{true}.$
$D(1) :\Leftrightarrow \text{true}.$
$D(x) :\Leftrightarrow \text{true}.$
$D(a[i]) :\Leftrightarrow D(i) \wedge 0 \leq i < \text{length}(a).$
$D(e_1 + e_2) :\Leftrightarrow D(e_1) \wedge D(e_2).$
$D(e_1 * e_2) :\Leftrightarrow D(e_1) \wedge D(e_2).$
$D(e_1/e_2) :\Leftrightarrow D(e_1) \wedge D(e_2) \wedge e_2 \neq 0.$
$D(\text{true}) :\Leftrightarrow \text{true}.$
$D(\text{false}) :\Leftrightarrow \text{true}.$
$D(\neg b) :\Leftrightarrow D(b).$
$D(b_1 \wedge b_2) :\Leftrightarrow D(b_1) \wedge D(b_2).$
$D(b_1 \vee b_2) :\Leftrightarrow D(b_1) \wedge D(b_2).$
$D(e_1 < e_2) :\Leftrightarrow D(e_1) \wedge D(e_2).$
$D(e_1 \leq e_2) :\Leftrightarrow D(e_1) \wedge D(e_2).$
$D(e_1 > e_2) :\Leftrightarrow D(e_1) \wedge D(e_2).$
$D(e_1 \geq e_2) :\Leftrightarrow D(e_1) \wedge D(e_2).$

Assumes that expressions have already been type-checked.

Abortion

Slight modification of existing rules.

$$\frac{\{P \wedge b \wedge D(b)\}\ c_1\ \{Q\} \quad \{P \wedge \neg b \wedge D(b)\}\ c_2\ \{Q\}}{\{P\}\ \textbf{if}\ b\ \textbf{then}\ c_1\ \textbf{else}\ c_2\ \{Q\}}$$

$$\frac{\{P \wedge b \wedge D(b)\}\ c\ \{Q\} \quad (P \wedge \neg b \wedge D(b)) \Rightarrow Q}{\{P\}\ \textbf{if}\ b\ \textbf{then}\ c\ \{Q\}}$$

$$\frac{\begin{array}{c} P \Rightarrow I \quad I \Rightarrow (T \in \mathbb{N} \wedge D(b)) \\ \{I \wedge b \wedge T = t\}\ c\ \{I \wedge T < t\} \quad (I \wedge \neg b) \Rightarrow Q \end{array}}{\{P\}\ \textbf{while}\ b\ \textbf{do}\ c\ \{Q\}}$$

Expressions must be defined in any context.

Abortion

Similar modifications of weakest preconditions.

$$
\begin{aligned}
&\mathrm{wp}(\textbf{abort}, Q) \Leftrightarrow \text{false}\\
&\mathrm{wp}(x := e, Q) \Leftrightarrow Q[e/x] \wedge D(e)\\
&\mathrm{wp}(\textbf{if } b \textbf{ then } c_1 \textbf{ else } c_2, Q) \Leftrightarrow\\
&\quad D(b) \wedge (b \Rightarrow \mathrm{wp}(c_1, Q)) \wedge (\neg b \Rightarrow \mathrm{wp}(c_2, Q))\\
&\mathrm{wp}(\textbf{if } b \textbf{ then } c, Q) \Leftrightarrow D(b) \wedge (b \Rightarrow \mathrm{wp}(c, Q)) \wedge (\neg b \Rightarrow Q)\\
&\mathrm{wp}(\textbf{while } b \textbf{ do } c, Q) \Leftrightarrow \exists i \in \mathbb{N} : L_i(Q)
\end{aligned}
$$

$$
\begin{aligned}
&L_0(Q) :\Leftrightarrow \text{false}\\
&L_{i+1}(Q) :\Leftrightarrow D(b) \wedge (\neg b \Rightarrow Q) \wedge (b \Rightarrow \mathrm{wp}(c, L_i(Q)))
\end{aligned}
$$

$\mathrm{wp}(c, Q)$ now makes sure that the execution of c does not abort but eventually terminates in a state in which Q holds.

Procedure Specifications

global F;
requires Pre;
ensures $Post$;
$o = p(i) \{ c \}$

- Specification of procedure $o = p(i)$.
  - Input parameter i, output parameter o.
    - A call has form $y = p(e)$ for expression e and variable y.
  - Set of global variables ("frame") F.
    - Those global variables that p may read/write (in addition to i, o).
    - Let f denote all variables in F.
  - Precondition Pre (may refer to i, f).
  - Postcondition $Post$ (may refer to i, f, f_0, o).
- Proof obligation

$$\{Pre \land i_0 = i \land f_0 = f\}\ c\ \{Post[i_0/i]\}$$

Procedure Calls

First let us give an alternative (equivalent) version of the assignment rule.

- Original:

$$\begin{array}{c}\{D(e) \land Q[e/x]\}\\ x := e\\ \{Q\}\end{array}$$

- Alternative:

$$\begin{array}{c}\{D(e) \land \forall x' : x' = e \Rightarrow Q[x'/x]\}\\ x := e\\ \{Q\}\end{array}$$

The new value of x is given name x' in the precondition.

Procedure Calls

From this, we can derive a rule for the correctness of procedure calls.

$$\begin{gathered}\{D(e) \wedge Pre[e/i] \wedge \\ \forall y', f' : Post[e/i, y'/o, f/f_0, f'/f] \Rightarrow Q[y'/y, f'/f]\} \\ y := p(e) \\ \{Q\}\end{gathered}$$

- $Pre[e/i]$ refers to the values of the actual argument e (rather than to the formal parameter i).
- y' and f' denote the values of the vars y, and f after the call.
- $Post[\ldots]$ refers to the argument values before and after the call.
- $Q[y'/y, f'/f]$ refers to the argument values after the call.

Modular reasoning: rule only relies on the *specification* of p, not on its implementation.

Corresponding Predicate Transformers

$$
\begin{array}{l}
\mathrm{wp}(y = p(e), Q) \Leftrightarrow \\
\quad D(e) \wedge Pre[e/i] \wedge \\
\quad \forall y', f' : \\
\quad\quad Post[e/i, y'/o, f/f_0, f'/f] \Rightarrow Q[y'/y, f'/f]
\end{array}
$$

$$
\begin{array}{l}
\mathrm{sp}(P, y = p(e)) \Leftrightarrow \\
\quad \exists y_0, f_0 : \\
\quad\quad P[y_0/y, f_0/f] \wedge Post[e[y_0/y, f_0/f]/i, y/o]
\end{array}
$$

Explicit naming of old/new values required.

Procedure Calls Example

- Procedure specification:

  global f
  requires $f \geq 0 \wedge i > 0$
  ensures $f_0 = f \cdot i + o \wedge 0 \leq o < i$
  $o = dividesF(i)$

- Procedure call:

  $\{f \geq 0 \wedge f = N \wedge b \geq 0\}$
  $y = dividesF(b+1)$
  $\{f \cdot (b+1) \leq N < (f+1) \cdot (b+1)\}$

- To be ultimately proved:

$$
\begin{array}{l}
f \geq 0 \wedge f = N \wedge b \geq 0 \Rightarrow \\
\quad D(b+1) \wedge f \geq 0 \wedge b+1 > 0 \wedge \\
\quad \forall y', f' : \\
\quad\quad f = f' \cdot (b+1) + y' \wedge 0 \leq y' < b+1 \Rightarrow \\
\quad\quad\quad f' \cdot (b+1) \leq N < (f'+1) \cdot (b+1)
\end{array}
$$

Not Yet Covered

- Primitive data types.
  - `int` values are actually finite precision integers.
- More data and control structures.
  - switch, do-while (easy); continue, break, return (more complicated).
  - Records can be handled similar to arrays.
- Recursion.
  - Procedures may not terminate due to recursive calls.
- Exceptions and Exception Handling.
  - Short discussion in the context of ESC/Java2 later.
- Pointers and Objects.
  - Here reasoning gets complicated.
- ...

The more features are covered, the more complicated reasoning becomes.